MODELS TO ILLUSTRATE THE FOUNDATIONS OF MATHEMATICS

BY

C. ELLIOTT

Price 2s. 6d. net

EDINBURGH

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CONTENTS.

CHAPTER II.—MULTIPLEXES.

CHAPTER III.—SPACES.

Summary of Chapter III.

CHAPTER IV.

CORRESPONDENCE OF OPERANDS TO FUNCTIONS.

CHAPTER V.—MULTIPLE CORRESPONDENCE.

MODELS TO ILLUSTRATE THE FOUNDATIONS OF MATHEMATICS.

INTRODUCTION.

There are no fixed and unalterable beginnings of mathematics. As progress is made in mathematical science, it becomes necessary from time to time to alter the presentation of the rudiments, and to view subjects which have found a place in elementary mathematics for centuries under a new aspect. Were this not to be done, the difficulties which a sudent meets with in attaining the point of view of advanced workers would continually increase, and would form a hindrance to further improvement.

The present writer considers that some existing difficulties may be overcome, at least in part, by the introduction of a new kind of practical work into schools. And though the reader may think that it is not feasible to do as proposed here, and make it form part of the groundwork of mathematical teaching, still it may be found of use as throwing fresh light upon the other methods employed to-day. The advantages or otherwise of any one view of mathematics can scarcely be realised until the attempt has been made to develop the subject from that one standpoint, so that, even if judged a failure, the endeavour may not be valueless to others.

This work supplements both some matters of everyday experience and certain aspects of ordinary elementary mathematics, and therefore is not an absolute novelty, but serves to emphasise some already existing features of mathematical teaching. On that account it is hoped that the following description may be intelligible even without the actual models, some examples of which, however, the reader is strongly advised to construct for himself.

They are intended to illustrate some modern views upon the Foundations of Mathematics, and to show that the "abstract" character of that subject does not forbid any attempt to bring

elementary teaching up to date in that direction. The importance of the ideas which can be illustrated by models in this way is fairly generally recognised, and it is hoped that the form in which they are here presented will make them more available for schools than they have hitherto been. The objection that such a method would be too abstract may be met by the fact that the ideas under discussion are visualised by the models, and that numerous examples are given, drawn from everyday experience.

A clear account of the development of pure or abstract mathematics will be found in a recent book, the *Fundamental Concepts of Algebra and Geometry*,* by Prof. J. W. Young, to which frequent references will be made. I have tried to make the description self-contained, so that possession of Prof. Young's work may not be absolutely necessary, but, for those who have not read that or some similar book, the remarks in the next three paragraphs will perhaps not convey much meaning, and may be passed over.

In that work it will be found that pure mathematics is regarded as consisting of attempts to deduce propositions "by the methods of Formal Logic" from postulates, or axioms, about terms of which the meaning is intentionally left undefined. These postulates or axioms are not looked upon as truths necessarily self-evident to the mind, nor as experimental facts, but merely as assumptions (p. 38). Calling such a series of logically connected propositions an abstract mathematical system, then mathematics as a whole is defined as consisting of all such systems together with all their concrete applications (p. 221).

This definition, however, the author points out, is not to be taken to imply that Formal Logic is the chief method of mathematical discovery. "*Imagination, geometric intuition, experimentation, analogies sometimes of the vaguest sort, and judicious guessing, these are the instruments continually employed in mathematical research*" (p. 221).

Yet the definition does seem to imply that a training in logic is absolutely necessary to comprehend the modern view of pure mathematics, and, as a consequence, that "*the points of view to be developed in these lectures, and the results reached, are not directly of use in elementary teaching*" (p. 7).

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* Macmillan, 1911, price 7s. net.

The present writer considers that, by viewing mathematics from a slightly different standpoint, less importance may be ascribed to abstract reasoning, and more to observation and description, and therefore the elements of modern pure mathematics become more accessible to beginners.

The kind of postulation referred to above is of course to be found only in modern works; in ancient text-books, such as that of Euclid, it exists in an obscure form which does not permit of accurate deductions according to modern standards. An examination of the nature of the modern form shows that what are postulated are classificatory relationships (a term which the models described later are intended to explain), and it is those which lend themselves to "formal reasoning."

The special point of view which has influenced the following treatment of elementary work is that from which mathematics is regarded as the science of classification. Such a statement as that the propositions of both Algebra and Geometry are among the implications of the same set of axioms (*Fundamental Concepts*, p. 183) we would translate by saying that the fundamental ideas of both can be illustrated by the same set of classificatory models. Or, again, the distinction drawn between pure and applied science (*Fundamental Concepts*, p. 54*) we would interpret by saying that the student of mathematics and physics must either be engaged in studying classification in general, that is, in arriving at the propositions of pure mathematics, or in obtaining by practical exercises the experimental data necessary for the application of his classificatory knowledge, or in making the application itself.

The pure mathematician is assumed to be really engaged in the investigation and description of classificatory relationships, and

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* "If we adopt the point of view of Peano and Russell, all *pure* mathematics is abstract. Any concrete representation of such an abstract science is then a branch of *applied* mathematics. Geometry, for example, as a branch of pure mathematics, consists, then, simply of the formal logical implications of a set of assumptions. Whenever we think of geometry as describing properties of the external world in which we live, we are thinking of a branch of applied mathematics in the same sense in which analytical mechanics is a branch of applied mathematics. We need not quibble over this distinction. The important thing is to recognise that *there exists an abstract science underlying any branch of mathematics, and that the study of this abstract science is essential to a clear understanding of the logical foundations.*"

hence the view is taken that models of classifications might play a part in the beginning of pure mathematical teaching, and be supplemented by tabulations of symbols, standing for classifications of the things symbolised. Such models are easy to construct, so easy that some readers may think they scarcely deserve the name of models, but, owing chiefly to the fact that mathematical language is somewhat redundant and defective, their explanation in simple terms is at present not free from difficulties.

No attempts at formal reasoning will be found therefore in the following pages, which are devoted solely to the explanation of those ideas relating to classification which play a prominent part in elementary mathematics, when looked at from the particular standpoint chosen. But although the deduction of propositions from a set of axioms, or unproved propositions, is avoided, yet the derivation of mathematical terms from one another has had some attention, in order to lay emphasis upon the necessity of beginning our explanations with a set of undefined terms.

Of the ideas to be explained, two of the most important, namely, that of a *correspondence or function*, and that of a *multiplex*, lend themselves readily to illustration by classificatory models, and therefore the opinion is set forth, though naturally with diffidence, that these ideas should, with some of their developments, be introduced at a very early stage, and form part of the groundwork of mathematical teaching. One difficulty met with is that it does not seem to be possible to employ such important words as Addition and Multiplication, the definition of which naturally follows that of Correspondence, throughout in only one sense, without somewhere coming in conflict with existing usage. The employment of those and some other words has, in course of time, been extended in a somewhat arbitrary and unsystematic manner, which puts a certain difficulty in the way of adhering to any one definition.

A model of a classification is intended to show those features which are important, and to omit all others; it is a set of things showing clearly those resemblances and differences alone, which are essential features of the type under discussion. The observation of likenesses and differences is a necessary preliminary to classification, and, since the differences with which we first become familiar are those of shape, size, colour, markings, and a few more, these are naturally to be utilised in the artificial production of classi-

fications for elementary teaching. Flat pieces of cardboard can be made to show a sufficient variety of differences from each other to serve for such artificial productions, and offer the advantage that scissors, and a paint-box, and brush are the only tools necessary.

It is, of course, not suggested that the usual school practical work in paper-folding, in the construction and measurement of diagrams and models, in the plotting of graphs, and so on, should be neglected, but only that the language and ideas based upon this work need to some extent to be revised, and, especially, the distinction between pure and applied mathematics placed in a clearer light. As already said, to adopt the plan suggested here is not to introduce absolute novelties, but to develop an already existing feature of mathematical teaching. For the classificatory ideas to be explained here are already studied in schools, but almost solely through applications, with which they are usually almost inextricably mixed. Therefore the beginner has at present to face two kinds of difficulties simultaneously, namely, those belonging to the body of classificatory ideas, and those inherent in some application. The aim of this book is to assist him by isolating the former from the latter.

This object requires more repetition of the ordinary elements of algebra and geometry than perhaps the reader would expect or desire in a work professing to deal with models, but it also involves a certain detachment from what are usually regarded as the main subjects of mathematical interest. For, although the mathematician is concerned with correspondences, and with multiplexes, and their related ideas, he is not exclusively concerned with the correspondences of numbers to numbers, nor with multiplexes of things differing in ordinary position. Thus the exclusive attention paid to numbers in school arithmetics, and to geometric points (in the common sense) in school geometries may actually disqualify to some extent such books from serving as an introduction to the ideas of modern mathematics.

In conclusion, it may be mentioned that a selection of the classificatory models referred to above was exhibited at the International Congress of Mathematicians held at Cambridge in 1912.

C. Elliott.

CHAPTER I.

THE MEANING OF CORRESPONDENCE.

Section 1.—The Words used in Mathematics.

Among other aspects mathematics may be viewed as a special language invented for special needs, and it possesses a set of words of its own, just as do music, or cricket, or botany.

Broadly speaking, the meaning of a new word may be explained in two ways, namely, by showing the thing or action for which it stands, or by the method used in an English dictionary, that of explaining the word by means of other words. If these other words are not understood, recourse must again be had to the dictionary. But, since the number of words in the dictionary is finite, we must, in the end, either use the words to be explained, in their own explanation, or be content to leave them unexplained, except by the first method, which might be called the laboratory or workshop method. This necessity of leaving some words undefined occurs, of course, in writings of every description, and not only in mathematics.

In order, therefore, to explain the words actually used in mathematics, the teacher must make a choice of words, the meaning of which he considers to be already known to his students, or of which he can easily demonstrate the meaning, and build his system of definitions upon these.

It has been customary for the two main branches of elementary mathematics, namely, Arithmetic and Algebra on the one hand and Geometry on the other, to be developed with distinct systems of words, and for no great attention to be paid to the list of words which it is necessary to leave undefined in each system.

Every-day experience, and practical exercises in school, are supposed to sufficiently demonstrate the meaning of the elementary terms employed in the two branches.

Now it is natural to ask whether mathematical science as a

whole could not be developed from but one set of words, and the question has been answered with some success as the result of modern research on the Foundations of Mathematics. It cannot, however, be expected that the old words employed in the branches treated separately will serve this new purpose. An entirely fresh set must be chosen, and the old words employed in Arithmetic or Geometry defined in terms of the new ones. It may, however, be expected that such a unification of the branches of mathematics will obviate the need for so many distinct terms by showing that the same concepts occur in what, from the older standpoint, were different departments, and that therefore the same words, and not distinct ones, may be used to express them.

Instead of such words as length, breadth, magnitude, direction, and the names of numbers, used in the elements of the old branches of mathematics, those of the new set (which, with their derived words, can be made to define the old ones) include such as class, belonging to a class, sub-class, correspondence, duplex, etc. The meanings of some of these words, which are used as undefined words in modern work, are considered in the paragraphs below, one object of this book being to show that these, too, like the older ones, can have their meaning illustrated both by school practical exercises, and by examples from every-day life.

The same remark applies to many of the words defined by means of them, which, on that account, could perhaps be equally well taken as undefined terms.

It may also be noticed that the teacher is under no compulsion to use *only* practical experience as a guide to the meaning of his undefined words, and *only* verbal explanations for his defined words. On the contrary, there seems to be no good reason why, in the beginning of mathematics, the former words should not be accompanied by verbal explanations, and the latter by practical illustrations. Nor is it to be expected that axioms and deductions should, in a school course, follow immediately upon the introduction of a few words. The importance of the fundamental ideas relating to classification is so great that it may be better to dwell upon their explanation and illustration, rather than to provide exercises in reasoning.

Section 2.—Some Undefined Words.

(*a*) The first we shall take is *respect*, or kind of difference. Various kinds of difference between things will already be known to the student, for example, differences in ordinary position, in size, shape, taste, hotness, etc., and school practical work in science consists largely in adding to their number. Among the differences a knowledge of which is gradually acquired are density, refractive index, specific heat, and many others. Therefore when, in what follows, bodies are spoken of as differing in one, two, three, or more respects, it will be assumed that a sufficiently clear meaning is attached to the word respect. The possibility of giving importance to the word has been brought about by the introduction of elementary practical science into schools.

(*b*) *Class.*—Things form a class when they agree in at least one respect or circumstance, and differ from each other in at least one respect. Thus, the chairs in this room form a class, because they agree in the circumstance that they are all chairs in the room, and they differ in position. To make clear the nature of what is meant by "class," we can construct model or artificial classes, the members of which agree in some given respect, as colour, and differ in some given respect, as shape. Pieces of cardboard properly coloured and shaped will serve this purpose. It is generally supposed that we are unable to distinguish different bodies at all, if they occupy the same position, and therefore, in whatever other respects the members of any class differ, they must differ in ordinary position as well. For example, the pieces of cardboard referred to must differ from each other in position, obviously, as well as in shape. For the sake of brevity in speaking of any class, the respect in which its members are alike will be referred to as A, and the respect in which they differ as D, so that, in the model suggested as an example, A is colour, and D is shape. It will be seen that the meaning of the word class above has been explained by making use of the word "respect," previously selected as an undefined term.

A peculiarity of our mental capabilities may be noticed while considering classes, namely, that we are able to group unlike things into an entirely arbitrary class, and remember the individuals thus placed together. The fact that this mental process has been applied

to them constitutes the respect A in which the members of the arbitrary class agree. The grouping is arbitrary in the sense that, though the person who made it can state at once what objects out of a number are, or are not, in the class, no mere examination of the objects would enable another person to do so, as he would be unable to detect any property A, serving to distinguish the members of the class from the rest.

(*c*) *Sub-class.*—One class is said to be a sub-class of another when all its members belong also to that other. Since they belong to it, they must agree in the respect A of the other, or they would not belong to it at all, but with regard to the respect D of the larger class it seems advisable to consider at least two cases. For the members of the sub-class may *differ* in D and agree in some third respect, which marks them off from the rest of the class, or they may *agree* in D, but differ in a third respect, D′. Both kinds of sub-class should be illustrated by artificial classes, containing sub-classes. As already said, the members of a class may be arbitrarily grouped together, and, similarly, a sub-class may be constructed by making an arbitrary selection of members from a class.

(*d*) *Classification.*—A class may have many sub-classes, and any sub-class may, in its turn, have its own sub-divisions. To SORT a class of things is to distribute them into sub-classes, and the resulting distribution is called a *classification* of the class with which we began. In illustration may be shown collections of various kinds, or such a case as the classification of families by nationality, town, street, and house, may be referred to. If it is to boys' drawing, modelling, carpentering, and counting that teachers look to supply a knowledge of the older undefined terms, it is perhaps boys' *collections* of things which may help to play the same part for the undefined terms now under consideration.

It has already been said, that one view which might be taken of mathematics is, that it is a special language invented for special needs. Of these special needs one of the most important is that of describing different kinds of classification. We might suppose a person to collect so many kinds of things, as to become virtually a collector of collections, a classifier of classifications. It is in this super-classification, or comparative study of classifications, that the foundations of mathematics are to be sought, and it is natural to

begin with trying to classify the simplest classifications, namely, single classes. Up to now, almost the only data for *classifying classes*, which we have considered, are the respects A, in which members agree, and the respects D, in which they differ. For example, several classes might be grouped together, merely from the circumstance that the members of each agree in *colour*, instead of A being colour in one case, hardness in another, size in another, and so forth. In later sections other ways of classifying classes are considered.

As we can place together things to form an *arbitrary* class, or to form arbitrary sub-classes of such a class, so, in general, we can form any kind of classification whatever of the given things in a purely arbitrary manner. The importance of this remark will be seen later, when the question of putting things in arbitrary correspondence with each other, and of arranging them in an arbitrary order, is considered.

*Section 3.—"Likeness" and "Position" of Things in a Class.**

The things which form a class may resemble and differ from each other in many ways besides the two ways, A and D, which are necessary for the existence of the class. Two members will be called *alike* or equal, or the same members, if they are alike in all respects, except D. With this meaning of "alike" it will be clear that all the members of a class might be "alike," or again, some "alike" in one way, and others "alike" in a different way, cases which can be illustrated by making artificial classes.

Two members of two *different* classes will be called "alike," or the same members, if they are alike in all respects, except the A's and D's of their respective classes. If they happen to be alike in

* In making an elementary classification of correspondences (Sec. 7), it is necessary to attach a clear meaning to the words "alike" and "same," and hence the introduction of this section. For example, one kind of correspondences consists of those having the "same" operands on both sides of the correspondence. It is not suggested, however, that the teacher should raise difficulties here which are not felt by his pupils. If the elementary classification of correspondences can be taught to beginners without entering into the meaning of "same" and "alike," then this course should be followed.

the D's as well, an important case which is considered later when dealing with correspondences, there is of course still more justification for calling two such members "alike," or the "same" members. In this sense, two classes may possess things in common, or overlap, and it may also happen that the one class is included in the other, that is, that every member which occurs in the one class occurs also in the other. Such cases, which can be illustrated by models, provide rough means of classifying classes, in addition to those already mentioned.

"Like" members in a single class differ only in their state as regards D, and the state of members of a class as regards D is called their "*position*" in the class. The term may be extended to classifications in general, because, like single classes, they provide as it were blank forms, the "*positions*" in which may be filled up by things which are all different, or which are of several kinds, or even which are all alike.

As in the case of most mathematical terms, difference of "position" is here borrowed from its common or workshop use, in which it means *one particular kind of difference*, with which every one is as familiar as with difference of taste or of colour. If a thorough revolution were to be made in the building up of mathematical language, the direction of such borrowings might in a sense have to be reversed, because the classificatory meaning of some words would perhaps then be the first with which the learner would become familiar, and it would be their use when applied to special classifications which might cause difficulty.

*Section 4.—The Meaning of Correspondence.**

If the respect D is the same for two classes, and the states of D are also the same, then the two classes are said to correspond

* The relations of classes in respect to their extension, that is, the inclusion, or overlapping, or exclusion of classes, might be regarded as forming the "foundations" of Logic in the same sense as they, and the relations hereafter considered, are here looked on as forming the "foundations" of Mathematics. At this point then we leave the ground common to Logic and Mathematics, if a distinction is made between them, and pass to a new relation of two classes, namely, their correspondence, the study of which is fairly characteristic of mathematics as commonly understood.

to each other. For example, in the one class A might be colour, and D shape, and in the othes class A might be a different colour, but the differences D might be the same differences of shape as occurred in the first class. Then the members of the first class would correspond to those of the second class, and, in this way, models can be made of the various kinds of correspondence, which will be considered later. Merely by using pieces of cardboard, properly coloured and shaped, we can represent such a correspondence as one of colours to colours, of shapes to shapes, of colours to markings made on the cardboard, and so forth. It may be noticed that such a model represents uniquely what it is intended to represent, and allows of but one interpretation. If it represents, say, a correspondence of colours to colours, then, to a colour on one side of the correspondence, only one colour is shown as corresponding to it on the other side, and no one can be in doubt as to which colour is meant.

The members of the two classes are said to have been classified and cross-classified, and their correspondence to each other is regarded (from the standpoint of this book) as arising through this classification and cross-classification.* In the method here adopted the phrase is considered to describe correctly the essential fact in all kinds of correspondence, and to be one which suggests the method of constructing artificial correspondences of an elementary kind as described above, which, in their turn, facilitate the classification of correspondences.

It is to be expected that, according to the nature of the things classified, and the considerations giving rise to the cross-classification, different names will be given to the correspondences. For example, one class may be said to *depend upon*, or *vary with*, or have an *association* or *correlation* with, or be a *function* of the other. Or all the members of the one class may be said to bear the *same relation* to their corresponding members in the other class.

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* The more usual procedure is to take "correspondence" as an *undefined term*. For example, y is said to be a one-valued function of x, if to every value of x there "corresponds" a unique value of y. This is satisfactory only if the student has already had many examples of correspondence in arithmetic and geometry without examining the idea of correspondence itself; that is to say, if he has sacrificed some of the advantages attaching to an early introduction to the idea.

And, with such general names for correspondences, we naturally find also general names, as *operand* and *variable*, to mean the things which correspond.

The circumstances surrounding particular cases of correspondence must not be allowed to obscure that simple feature (classification and cross-classification), which is enough by itself to permit of the application of a valuable range of mathematical ideas. The extensive use of graphs has familiarised the general public with the far-reaching importance of the idea of functionality, and schools might now reflect the belief in a more fundamental way than merely by the introduction of some work on curves, and the plotting of experimental results. For the idea to play a really important part in elementary work, it is not so much those correspondences of numbers to numbers to which the word function was applied in the seventeenth century, or those others, found experimentally, which are to be studied, as rather correspondence itself. In order to make clear the sense in which a classification and cross-classification is regarded as underlying all instances of correspondence, some common examples are considered below.

In so far as some of these illustrations involve ideas of order, space, or number, they must, of course, be regarded as out of their proper position in a logical development of the Foundations of Mathematics. They should be mentioned only after the ideas of order, space, and so forth, have been explained through classificatory models, as is done in later sections.

Section 5.—Illustrations of Correspondence.

1. The correspondence of words of the *same meaning* in two languages, such a correspondence, for example, as is tabulated in an English-French dictionary. The correspondence might be described by saying that the words have been classed as English or French, and then cross-classified by meaning. The same fact could be conveyed by saying that the English words are a *function* of the French ones, or that they *depend on* them, or *vary with* them. In this example the things taking part in the correspondence, that is, the "*operands*" or "*variables*" happen to be words, while, in the correspondences which it has been usual to study first

in mathematics, they are numbers, and, in the examples below, they are bodies, or geometric points, or persons, etc.

The use of the word "*operand*" is explained by *some* correspondences being due to the fact that each member of the one class has been derived from the corresponding member of the other class by effecting some particular change, or process, or "*operation*" on it.

In that case the "operation" provides the necessary grounds for cross-classification, and, in the example above, the "operation" consists in translating English into French.

As regards the word "*variable*," it will be clear that, if two things which are equal in some respect undergo changes or "vary" in that respect, while still remaining equal to each other, then, by this fact, a correspondence is provided between the states of the one thing and those of the other, and therefore it is natural to use the word "variable" for a state of change which takes part in such a correspondence.

It will be seen that both "operand" and "variable" strictly speaking imply, though in different ways, that *change* has played a part in giving rise to the correspondence, and therefore it might be argued that their use as general terms for things taking part in a correspondence is open to objection, because changes in that sense are not an essential feature of all correspondences. For example, change plays no part in providing the correspondence of words to words discussed above, since the English words cannot be transformed into each other, nor into French words.*

2. The correspondences of amounts of two different substances having the *same value*, as copper and silver. In this example the cross-classification is by value, and the correspondence might also be described by saying that the pieces of copper are a function of, or depend upon, or vary with, the pieces of silver.

3. The correspondences between persons expressed by saying that one is the father, or husband, or grandfather, and so forth, of another. Supposing that the correspondence "*father*" is to be

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* In order to free the word variable from its connection with the idea of change, in J. W. Young's *Fundamental Concepts*, the term is reserved for "a symbol which represents *any one* of a class of elements" (p. 192), and which, therefore, may represent *any* operand on one side of a correspondence. Any particular operand represented by a variable x is called a *value* of the variable.

tabulated, then to each name of a person on the one side of the correspondence, we have, on the other side, the name of the father of that person. In this example the operands are not words, or pieces of metal, but persons, namely, sons on one side of the correspondence, and fathers on the other, and the persons on the one side are a function of, or depend upon, or vary with those on the other.

4. The correspondence of subject to object of a transitive verb. Suppose that the subject is some person, the verb is "spells" and the object is some word. Then, if the function is tabulated, we have, on the one side a list of names of persons, and, on the other, to each person a word which he has spelt.

5. The correspondence of a point in a district to a point in a map of the district; or of a point of an object to its image, produced by a lens on a screen. In this example the operands are geometric points, and those of the map are a function of, or depend upon, or vary with those in the district mapped.

6. The correspondence of the states of two changes given by the fact that the states occur simultaneously. For example, the correspondence of weights hung on a spring, to the extensions of the spring produced by the weights; or again of a boy's weight to his height at the same age. It is important to notice that the grounds for cross-classification here, that is, the reason for placing together two states of change, is simply, that they occur at the same moment, and this is true for almost all the functions investigated experimentally in the laboratory. Since we are concerned with two changes, the word *variable* can be used for a thing taking part in the correspondence with better reason than in some of the other examples.

7. If we suppose that the members of a class can have their positions altered, by interchanging members, then there is a correspondence between any member, and that member the place of which it takes. The two classes which correspond are, in this instance, the original class, and the class after the exchange of positions has taken place. Such a correspondence is called a "*permutation.*"* Anything which appears on one side of the correspondence appears also on the other, only in a different position,

* *V.* Burnside, *Theory of Groups.* Camb. Univ. Press.

and therefore these correspondences are interesting as an example of those functions in which the "same" set of things appears on both sides. The meaning to be ascribed to "same" was explained in Section 3.

8. The correspondence to each other of properties existing in one body. Supposing a body has a certain weight and a certain volume, it may be moved about without these two properties altering. The different positions of the body furnish a class of weights, and a class of volumes, and the association of a weight with a volume in the body in any one position furnishes a ground for their cross-classification; so that, in this illustration, the correspondence consists of weights on the one side, and volumes on the other. As pointed out in Ostwald's *Fundamental Principles of Chemistry*,* it is this unalterable association of certain properties which is summed up in the word "*body*" itself, and the use of "*body*" renders that of "matter" superfluous. It will be noticed that all the weights of the class of weights are the "same," and all the volumes of the class of volumes are the "same," in the sense in which "*same*" was explained in Section 3, so that this function is of a kind different from the others considered above.

9. Correspondences of numbers to numbers.

Before considering the illustration below, it may be well to explain how the meaning of a classificatory term such as "function" is extended from its simple form so as to embrace more difficult cases.

We have seen that it is convenient to have a general term, as "*operand*," for a thing taking part in a correspondence, or, in a still wider sense, for anything taking part in a classification.

Now, beginning with the idea of a single class, it may be said that we pass from it to that of a classification by imagining that classes have taken the place of operands in the single class. This kind of mental substitution can be repeated without limit, and leads us on to the most complicated classifications. But, having learnt from models the meaning of a correspondence, namely, that it is a classificatory idea like the idea of class, it is natural to ask whether that mental step which may be regarded as leading from a class to a classification cannot be imitated in the case of corre-

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* 1909, p. 9, Eng. trans.

spondences. The imitation consists in substituting classes or functions for the operands. If classes are substituted, then the correspondence becomes one of classes to classes, and, if functions are substituted, then such a correspondence as one of operands to functions, or of functions to functions. And these cases are not unimportant subjects of enquiry, but matters of vital interest in elementary mathematics, so that it is worth while to consider a line of thought which connects them with the simple ideas of class and of correspondence, which can easily be explained by models. The passage from a simple classificatory idea to more complicated forms of it is here supposed to be made by substituting for the operands of the simple form either classificatory ideas already understood, or even the idea itself which is to be developed into a more complicated form. We have an example of the first kind of substitution when the *classification of functions* is considered, and of the second when the *correspondence of functions to functions* is considered.

It appears that, for a word to permit of just this kind of extension of its meaning, it is essential that it should be a classificatory word, for it is those alone which have a meaning independent of the nature of their operands.

If the original conception can be satisfactorily represented by a model, and also those which are, so to speak, inserted into the original conception, then we can rest assured that the final result could also be represented by a model, though probably no useful purpose would be served by constructing such an intricate one.

For example, if the correspondence of operands to functions is to be considered, as it is proposed to do in the illustration below, the general features of such correspondences could, if necessary, be studied by means of models, as already done for the simpler idea of correspondence. But, as the chief use of those models is to demonstrate the part played by classification and cross-classification, and to facilitate the elementary classification of correspondences, they may in such a case as this give place to the more convenient representation of correspondences by symbols, to be explained later on.

The operands of the correspondence now to be considered are numbers, and the type of correspondence is, as already said, that of a correspondence of operands to functions. It may be said that

it is usual to begin arithmetic with correspondences of this type. For, in an addition table of numbers, we have, to each number, a *correspondence of numbers* obtained by adding that number to the different numbers in turn; and in a multiplication table of numbers we have to each number a *correspondence* of *numbers* obtained by multiplying the different numbers in turn by that number. So that, on the one side of such a correspondence, we have simply numbers, but, on the other side, correspondence of numbers to numbers.* For example, to the number 3 belongs, in the addition table, the correspondence or operation $+3$, and, in the multiplication table, the correspondence or operation $\times 3$.

Since the meaning of numbers, and of addition and multiplication, has not yet been considered, the illustration given above must be regarded as out of its proper place in a logical development of the Foundations of Mathematics; that is, a development in which only those words are used which have been explained in terms of preceding words, or which were included in the list of words to be left undefined.

10. The correspondences of points to points called vectors.

If from any point we proceed in a given direction for a given distance, we arrive at another point, which thus corresponds to the first.

A correspondence of this kind is called a vector, and is sometimes defined as "the operation of carrying a tracing point in a certain direction for a certain distance." A definition which is perhaps a little misleading, as it is not the carrying of the tracing point, but the correspondence to which the carrying gives rise, which is of importance, and lends itself to mathematical treatment. Since ideas of order, distance, and direction are intimately involved in that of a vector, perhaps the latter cannot be expected to be clear until the former have been explained.

The operands of the correspondence are geometric points, and, since any point may be either the origin or the termination of a given vector, the correspondence is of that kind where the "same"

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* Those familiar with the technical meaning of the word "Group" (*v.* Chap. IV.) will perceive that, from the present standpoint, any Group is a correspondence of operands to functions, though every instance of the latter is not a Group.

operands appear on both sides, like the "permutations" considered above.

Section 6.—Symbols.

It has already been pointed out that, whatever the mental process may be, it is a fact that we can group things into purely arbitrary classes. The nature of a correspondence, therefore, leads us to expect that the members of two classes can also be put into quite arbitrary correspondence with each other. The things which have been thus arbitrarily associated with the members of another class might be called *symbols* for them, a term which is, however, often reserved for a special kind of symbol, namely, signs on paper.

Using the general meaning of symbol, names must be regarded as symbols for the things they stand for, since there is clearly no necessary connection, or mankind would have but one language. Other examples of symbols are the Arabic digits, which are symbols for numbers; the symbols for the chemical elements; the letters of the alphabet, which are symbols for sounds, and are also used as symbols in many other ways; the flag positions of the semaphore code, which are symbols for the letters of the alphabet.

If the states of the various respects in which things differ are represented by symbols, then a body is represented by a combination of symbols, one for each of its "properties." Therefore, as a classified body is represented by a combination of symbols, a classification of bodies is represented by a table or set of such combinations. Thus *tables* of combinations of symbols may take the place of models as a means of studying *different kinds of classifications*. Naturally, if employed for that purpose, the symbols represent only those properties of the bodies which are used in the classification.

Frequently important features of a classification may be represented not entirely by combinations of symbols, but also by the relative position of the symbols on paper, as when the symbols for corresponding operands of a function are written *opposite* each other. This plan is, however, open to the danger of suggesting, in some cases, that geometrical relationships exist in the common sense, whereas only classificatory relationships are intended to be shown; a danger especially great in the case of the structural formulae of chemistry.

Since the number of *combinations* of symbols which can be made from the symbols of several alphabets is much larger than the total number of *separate* symbols, it may be expected that classifications of bodies will enable us to economise symbols, that is, to use fewer symbols to represent the bodies than if a separate symbol were to be assigned to each. Thus, not only do symbols facilitate the study of classifications in general, but also a classification of things facilitates their representation by symbols.

If a classificatory idea, such as that of a correspondence, is represented by symbols, then, if the idea is extended in the way already explained, the extension is shown by new sets of symbols filling the places of those for the old operands.

The symbolic representation of the two simple classificatory ideas already discussed, namely, *class*, and *correspondence*, may be begun by taking that of classes.

The problem, then, is how to represent a member of a class so as not only to show to which particular class it belongs, but also to distinguish it from the other members of that class. Two ways of doing this present themselves: *firstly*, to use a separate alphabet for each class, or, *secondly*, to use but two alphabets, one to distinguish *members* of any class, and the other, the symbols of which must be combined with those of the first, to show which particular *class* is being represented.

A correspondence, since it consists of two classes, can be tabulated by writing the symbols for these classes in two columns, and placing opposite each other the symbols for the two members which correspond. If the second method above for representing members of a class is employed, then it is natural to use the *same* position symbols for any *two corresponding* members, which are then distinguished only by the class symbols. For example,

$$\begin{array}{ll} A_a & B_a \\ A_b & B_b \\ A_c & B_c \\ A_d & B_d\,. \end{array}$$

If the correspondence is of the kind where the "same" operands which appear on one side appear also on the other, as, for example, a permutation, then, instead of a double column of symbols, we may simply write the symbols in the order in which they correspond.

Thus, in the case of a permutation, an order, as *a b c d*, means that *b* has taken the place of *a*, *c* of *b*, *d* of *c*, and *a* of *d*.

The operands on one side of a correspondence are sometimes called a "quantity," so that a table such as that above represents the correspondence of the quantity A to the quantity B.*

The fact that a correspondence, or function f_1, exists between two classes or quantities, A and B, is expressed by writing $A = f_1 B$.† This is an instance of "syncopated algebra,"‡ for the sign = merely stands for the word *is*, and f_1 stands for the words "a *function* f_1 *of*." The symbol f_1 may also be described as the "operator" which converts an operand of B into the corresponding operand of A, and this may be indicated by writing $\frac{A}{B} = f_1$. This may be read: "The operation which converts an operand of B into the corresponding operand of A is f_1."

Any of the examples of correspondence already considered could, if necessary, be symbolically represented in this way. If f_1 stands for the correspondence of English to French words, then B will stand for the class or quantity of English words, and A for the class or quantity of French words.

In Algebra, the notation is used in such cases as $A = \text{sine } B$, $A = \log B$, $A = \tan B$, but, as already said, it is usual to begin *Arithmetic* with correspondences of operands to functions, instead of with the simpler idea of a correspondence of operands to operands. If A and B are classes of functions, then f_1 is a

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* There is some convenience in using capital letters for quantities, and small letters for operands. As already said, Prof. J. W. Young employs "variable" to mean a symbol standing for *any* operand of a quantity. If A is regarded as a variable, and represents the operand A_c above, then A_c is its "value" at that position.

† Unless the operands concerned also form a "Group" (*v.* Chap. IV.), the equation cannot be put in the forms $A - fB = 0$, or $A \div fB = 1$.

‡ It is usual to distinguish three stages of development of algebra, namely, *rhetorical* algebra, *syncopated*, and *symbolic*. In the first, ordinary language is used, in the second, symbols take the place of words, but merely as abbreviations, in the third there is no obvious connection with ordinary language, algebra forming a kind of language by itself. For at least three centuries, however, mathematicians have made use of all three methods according as the needs of the work on which they were engaged made one style of writing more convenient than the others.

correspondence of functions to functions, and if only B consists of functions, then f_1 is a correspondence of operands to functions. Thus, the simple notation $A = f_1 B$ finds no useful employment in the beginning of Arithmetic, as taught at present.

Elementary Geometry, as at present taught, shows the striking difference from arithmetic and algebra that symbolic methods are little employed, this being due to the fact that the idea of correspondence is made little use of. But, so far as the *elementary symbolism of correspondence* is concerned, geometry is perhaps better adapted than arithmetic for illustrating its main features, and such a correspondence as that of two points, got by drawing a line in a certain direction, and for a certain distance from one, and so arriving at the other (vectors), involves no more unexplained geometrical words than are already freely used in school geometries. In this instance we should be dealing with points, symbolised by letters of the alphabet, as operands, but, in a numerical correspondence, it would be difficult to avoid the use of the Arabic notation for numbers, which would, therefore, logically demand an explanation first of all.

It will be seen that, if the idea of correspondence is looked upon as one of fundamental importance, and made the basis of mathematical teaching, then we are almost obliged to make some elementary Algebra, if it may be called such, precede Arithmetic. In fact, the question might even be raised whether it is easier to teach algebra as a generalisation of arithmetic, and therefore following it, or arithmetic as a special case of algebra, which might be taken first (*v.* Summary of Chap. I.).

Section 7.—Classification of Correspondences.

Naturally only the broad divisions into which correspondences fall can be considered here, namely, those which can easily be demonstrated by models. Apart from the obvious plan of classifying correspondences according to the nature of the things which correspond, two main ways are suggested by the remarks already made (Sec. 3) upon the different kinds of classes. We may consider *firstly*, whether the members of a quantity are, or are not, all different from each other, and, *secondly*, whether the members which appear in one of the two quantities do, or do not, appear in

the other. These two ways of differentiating between correspondences are independent of each other, so that we can classify correspondences in one of the ways, and then subdivide a class by the other way.

Unfortunately, the question of what *names* to give to these elementary divisions is one of some difficulty, for several names appear to be available for some important cases, and none at all for others. For example, if we consider either of the two quantities, and find that its members are all different from each other, and that the same is true of the other quantity, then the correspondence is called a unique, or single-valued, function. But how shall we describe the important case where the "same" operands which appear on the one side appear also on the other?

Perhaps the simplest plan may be to divide correspondences into two classes, and call one *unlike-class* correspondences, and the other *twin-class* correspondences. The correspondence tabulated in an English-French dictionary is an "*unlike-class*," because the two sides of the correspondence are different, but a *permutation* is a "*twin-class*," because every operand which appears on the one side appears also on the other. Of the correspondences tabulated below,

(1), (2), and (3) are "unlike-class,"
(4), (5), (6), (7), (8), are "twin-class."

(1)		(2)		(3)	
a	l	a	b	a	b
b	m	a	b	b	c
c	n	a	b	c	d
d	p	a	b	d	e
e	q	a	b	e	f

(4)		(5)		(6)		(7)		(8)	
a	b	a	b	a	b	a	a	a	b
b	c	b	a	b	c	b	b	b	c
c	d	c	d	c	a	c	c	b	c
d	e	d	c			d	d	c	a
e	a					e	e	c	b
								c	a

We may now subdivide each of these two kinds of corre-

spondence, the unlike class and the twin-class, by considering whether the operands of a quantity are all different, or whether some are alike.

For example, in (1) they are all different, the case which we meet with in an English-French dictionary.

In (2) all the operands on one side are alike of one kind, and are, therefore, represented by the same symbol, a, and those on the other side are all alike of another kind, b. These correspondences which the word "body" serves to remind us of are of this kind.

In (3) the operands are all different, but this correspondence is almost a twin-class correspondence, because, with the exception of a and f, each operand appears on both sides. It will be found that it is convenient to call a correspondence of this type an "*open order*," or an order which has a *first* member a, and a *last* member f. The words *order*, *first*, and *last* are for the moment introduced without explanation.

Turning now to the twin-class correspondences, (4) is an example of a "closed order" or "*cycle*." The correspondence gives its operands an order, but no operand can be selected as a "first" or "last" member of the order. Experience in making models of twin-class correspondences, or in tabulating them, will show that a twin-class correspondence must, if the operands are all different, either consist of one cycle, or be made up of several. Thus (5) consists of two small cycles, each of which is of the kind called a transposition, a cycle of but two operands. (6) represents a cycle of three operands, which might be called a treposition. In (7) we have a correspondence which might be regarded as consisting of cycles of but one operand. Each operand on the one side corresponds to the "same" operand on the other. A correspondence of this kind is known as "*one*" or as the "*identical operation*" which "transforms each operand into itself." (8) is an example of a twin-class correspondence where the operands of a quantity are not all different.

Section 8.—Change and Order.

The fact that the operands of a cycle are arranged in an order naturally suggests the question whether a correspondence of this kind may not be the essential feature in all instances of *order*, just as classification and cross-classification appear to be an essential

feature of all instances of *correspondence.* Such an explanation of the meaning of order offers an attractive simplicity, but it must be admitted that the subject is surrounded by obscurities.

One apparent difficulty is that a *change* gives an order to its states, and it may reasonably be asked, in what does the classification and cross-classification consist here, if by *order* is meant a special kind of *correspondence*? It scarcely appears satisfactory to say that we mentally analyse each state of the change into *two*, a "beginning" and an "end," and that one side of the correspondence consists of "beginnings" and the other of "ends." The words "beginning," and "end," appear to require for their definition the very idea of order which we are trying to explain.

If we brush aside these difficulties, and assume that orders can be represented by models in the same sense as other correspondences can, then the fact that we can arrange things in an arbitrary correspondence, furnishes an explanation of the fact that we can arrange the members of a given class in an *arbitrary order.**

In J. W. Young's *Fundamental Concepts of Algebra and Geometry*, the plan is adopted of laying down postulates (suggested by Prof. E. V. Huntingdon) about an *undefined relation* $<$ between the members of a class.† And these postulates amount to a definition of what is meant when we say that the members of that class have an order. But this plan, while lending itself to formal reasoning, cuts off in a sense the idea of *order* from the main body of classificatory ideas.

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* This suggests in turn a quantitative test of the problem under discussion. If we compare the capacities of persons to remember an arbitrary correspondence with their capacities to remember an arbitrary order, it would seem that these two capacities should be equal, if the mental processes are the same in both cases. I do not know whether statistics of such experiments have settled the point, and, even if they have, the bearing of the result upon the question at issue might still be open to dispute.

† "The expression $a=b$ (read : a 'equals' or 'is the same as' b) indicates that the elements a, b, in question may be interchanged in the discussion. The expression $a \neq b$ (read : a 'is distinct' or 'is different from' b) indicates that a and b may not be interchanged. The relation $<$ is then characterised by the following *fundamental assumptions* :—

"Given a class C and a relation $<$; let a, b, c be any elements of C.

O_1. If $a \neq b$, then either $a<b$ or $b<a$.

O_2. If $a<b$, then $a \neq b$.

O_3. If $a<b$ and $b<c$, then $a<c$."

It seems like an admission that, in *change*, we meet with some obscure element which, with our present range of ideas and language, eludes description; by "our present range" is meant those ideas which can be illustrated by classificatory models.

In order to preserve the harmony of this book, and, make the utmost use of the idea of correspondence, it will be assumed in what follows that the words *cycle* of operands, and closed *order* of operands, are strictly equivalent, and that operands cannot give the one without giving the other. In other words, it will be assumed that the idea of order is inseparably linked with that of twin-class correspondence.

Section 9.—Direct and Inverse Correspondence.

If one class corresponds to another, and this correspondence is called a "*direct*" one, then the correspondence of the latter class to the former is called the "*inverse*" of it.

Many functions, or classes of functions, have special names to distinguish "direct" from "inverse." Thus, for example, the passive voice of a transitive verb is the inverse of the active voice.

A French-English dictionary tabulates the inverse of an English-French dictionary.

The correspondence "father" is the inverse of "son," "husband" of "wife."

To add one number to another is clearly the same thing as applying to that other a certain correspondence or operator, which has already been referred to above merely as an example of correspondence. To subtract the same number is to apply the inverse of that operator. Similarly, to multiply one number by another is the same thing as applying to the former a certain correspondence, and to divide by the second number is to apply the inverse of that correspondence.

In the case of *orders*, the comparative degree of one adjective is often used for direct correspondence, and the comparative degree of another adjective for the inverse. It will be remembered that we have chosen to regard orders as merely an illustration of a kind of twin-class correspondence, or of a correspondence which is nearly twin-class, in the case of "open" orders. Thus, "smaller than" is

the inverse of "greater than"; "harder than" the inverse of "softer than"; "earlier than" the inverse of "later than."

Or again, a verb as "*precedes*" can be used for the direct, and another verb, as "*succeeds*" for the inverse.

It may be noticed that the inverse of "*one*," and of transpositions, is the same as the direct form.

Another way of speaking of direct and inverse, as regards *orders*, is to call the direct correspondence one "*direction*," and the inverse the *opposite* "*direction*" in the order. The things which *succeed* one given thing in an order, but which *preceed* another thing are said to lie "*between*" the two things. It may be noticed that we are here using common terms of geometry, namely, "direction" and "between," in a purely classificatory sense, as was done before with the word "position."

As it is convenient to use separate names for the direct and inverse of a correspondence, so also for the two sides of a correspondence. If any member of the one class is called an independent variable, then any member of the other class is called the dependent variable. Sometimes the members of the one class are called the "argument" and those which correspond to them the "function."

It is of interest to notice that to learn by heart the direct form of a correspondence is by no means the same as learning by heart the inverse form. For example, we may be able to give at once the English word corresponding to a French one, but find it much more difficult to give the French word corresponding to a given English one. When the correspondence is an order, this is still more conspicuous; for example, the ease with which we remember the alphabet in direct order, may be compared with the difficulty of saying it backwards.

Every one is familiar also with the difficulty of beginning an order with some other member than the one with which he is accustomed to begin. Many persons, if asked to continue the alphabet from some letter as *m*, would find it necessary to go through the part preceding *m* first.

The difference in difficulty of working out an "inverse" correspondence as compared with the "direct" form is partly responsible for the difference between elementary Algebra and Arithmetic. To use the common notation, we may say that most arithmetical

problems are of the type $X = f_1 a$, where X is the answer sought, and a the given data, and most algebraical problems are of the type $a = f_1 x$; or that the answers to arithmetical questions are the data of algebraical questions.

If attention is paid to the correspondences of functions to functions, instead of operands to operands, then an obvious example of such a correspondence is that of the direct form of any function to its inverse. It will be found that this correspondence plays an important part in the explanation of the common notation for the addition and multiplication of numbers (*v.* Chap. IV., Sec. 17, the "Inversor").

*Section 10.—Numbers.**

Correspondence provides a simple mode of classifying classes, because all classes which can be put in correspondence with a given class may be grouped together.

All classes which can be placed in one-to-one correspondence with each other are said to have the same *number* of things. By a number such as five is therefore meant simply a class of classes.

Numbers are distinguished amongst other ways as *finite* and *infinite*, but only the former kind will be considered here, and, of them, only "rational" numbers.

Any correspondence of classes to classes is also a correspondence of numbers to numbers, as every class has some number; and numerical functions, *i.e.*, *correspondences of numbers to numbers*, all fall under one or other of the types of correspondence already discussed. The "operation" of addition of one class to another to give a third, mentioned in Section 5, is therefore an example of a correspondence of numbers to numbers. By addition of two classes is meant their arbitrary union to form a larger class, and therefore the primary meaning of the word relates to classes, and a further mental step is necessary to make it refer to numbers.

If a clear meaning is attached to the twin-class correspondence "ONE" and to "*addition*" of correspondences (*v.* Chap. I., Sec. 13) then the whole numbers can be looked upon from a different point of view, which however also involves the idea of correspondence.

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* *v.* J. W. Young, *Fundamental Concepts of Algebra and Geometry*, p. 60.

We can regard them as *functions* (two times, three times, etc.) obtained by the *addition* of *one* to itself.

Perhaps, having regard to the way in which addition of correspondence is defined (Sec. 13) it might be argued that these two modes of considering numbers are practically identical. The second way, however, possesses the advantage that the *inverses* of the functions (half, third, fourth, etc.) naturally present themselves for consideration, and have an equal claim to be looked upon as numbers, while, on the first plan, their introduction has a somewhat artificial character. For the same reason, namely, that numbers can be looked upon as functions, it is natural to consider their *multiplication*, and the word multiplication can be used in the same sense here as in regard to all other correspondences (*v.* Sec. 13, Chap. I.).

Section 11.—Relative Position.

By position in an *un*ordered class is meant simply the state of D with which a member of the class is combined. But in an *ordered* class we have to consider what is called "*relative position,*" by which is meant the statement as to whether a member is on the direct or inverse side of a given member. We have seen that to make such a statement the comparative degree of an adjective is often used. In that case we have made a "*qualitative*" statement of the relative position. If, in addition, we can give the number of correspondence lying "*between*" the two members under discussion, the statement becomes a "QUANTITATIVE" one.

The number of separate correspondences between the two members is called their "*distance*" apart in the order, and, as will be seen later, it might also be regarded as a "*power*" of the correspondence. The reader must remember that the words "closed order" and "cycle" of operands are in this book regarded as entirely equivalent in meaning. "*Power*" is a word from the language of Algebra which has the same meaning as *distance* in the language of Geometry, provided that "order" and "space" have the classificatory meanings adopted in this book. "*Distance,*" like "*position,*" is of course borrowed from its everyday use, where its meaning is restricted to a particular correspondence.

As a guide to the building up of new classificatory terms from

old ones, the method of supposing classifications to take the place of operands has already been referred to. In that way we pass from the idea of a class to that of a classification. Another clue to the origin of the new words is to be found in the idea of operands being *common to* two or more classifications.

Thus, for example, we have seen that the two quantities of a function may (in a certain sense) possess operands in common, and, if all the operands are possessed in common, the function is a twin-class one.

If two separate functions have a quantity in common, then, as will be seen later, this case is regarded as giving rise to the ideas of multiplication and addition of correspondences.

If the first and last operands of an "open order" occur as *two successive operands* in another order, then the operands of the first order are *inserted between* the two operands of the second. Since those two operands were at unit distance apart, it follows that the possession of operands in common by two twin-class correspondences may oblige us to consider *fractions of a unit distance.*

For example, if two operands are inserted between the two at unit distance, then the "distance" from one inserted operand to the other must be looked upon as a third. If only one operand is inserted, then its distance from those on either side is a half. The important thing to notice is that in order to consider *fractional distances* at all, at least two kinds of correspondence or operation are necessary, one of which serves to mark unit distances.

Section 12.—Counting and Measurement.

Besides giving the "distance" between two things in an order, we might give the number of things between them, or again in an open order we might give the number of things which precede a given thing, or the number which follow it. The number of things which precede a thing, including the thing itself, might be called its "*count,*" and, if the members of any class are arranged in an order, the "count" of the last is therefore the number of members in the class. Now, the numbers are arranged in an order beginning with one by repetition of the operation of including the class one. And in *this* order the "*count*" of a number has the same name as the number itself. Therefore to "count" the number of things in any

class is to put them into one-to-one correspondence with this order of *numbers*, and then the last *number* reached gives what is required, the "count" of that number, which is also the number of things in the class to be counted.

The amount of a distance is got by counting the number of correspondences between the two given things, each correspondence being called a "*unit*." But it is perhaps usual to include in the idea of "MEASUREMENT" not only the counting, but also the previous setting up of the particular correspondence employed. To speak of *the* distance between two things is meaningless, unless we have in mind some particular unit, by repetition of which, starting from one thing, we arrive at the other. We assume for the moment that there is an integral number of correspondences between the two things. As already said "distance" is essentially the same idea as "power" of an operation, and unit distance the same as unit power. To anticipate matters, it may be said that to *measure* a distance is very like the problem of finding a logarithm, the unit distance playing the part of the base of the logarithms, since, from the present point of view, the question is, to what power must the unit operation be raised to give the distance?

Section 13.—Multiplication and Addition of Correspondences.

Two classes which *correspond to* the same class correspond to each other, a statement which may be looked upon as the modern form of Euclid's axiom that things which are "equal to" the same thing are "equal" to each other. This relationship of the *third correspondence to the other two* is of a very simple and obvious character, just as is that of the "sum" of two *classes* to its addends, and it might naturally be supposed therefore that there would be one universally accepted name for it.

It might also be expected that the third correspondence would be called either the "sum" or the "product" of the other two. Unfortunately, in practice, it is called the sum in the case of a number of functions, but the product in the case of others. For the sake of clearness it will always be called the *product* here, partly because there are strong reasons for reserving the word "sum" for another almost equally simple relationship of one function to two others.

The *latter* relationship may be explained as follows: Suppose the two functions which have a class in common to be of the kind where single operands on the one side correspond to classes of operands on the other, and let the class of single operands be the one which they have in common. The table below shows a function of the kind under discussion. To a single operand as b on the one side corresponds a set of operands, namely, i, j, k on the other side.

$$a \; . \; . \; . \; . \; . \; . \left\{ \begin{matrix} e \\ f \\ g \end{matrix} \right.$$

$$b \; . \; . \; . \; . \; . \; . \left\{ \begin{matrix} i \\ j \\ k \end{matrix} \right.$$

$$c \; . \; . \; . \; . \; . \; . \left\{ \begin{matrix} l \\ m \\ n \end{matrix} \right.$$

The second function is supposed to be of the same kind, but, to the operand b, corresponds some different set, as R, S, T, U. Then an obvious way of deriving a new correspondence is to add together each pair of corresponding classes (*e.g.*, i, j, k and R, S, T, U), and, since each new *class* is called the sum of its addends, it is natural to call the new *function* the sum of the two others. Moreover, since two classes can be added even if they contain but *one* member each, it is clear that *any* two functions which have a class in common can be added, and it is not necessary to consider that they are of the kind just considered, which were discussed merely in order to show that it is quite *natural* to use the word sum in this connection.

This is further shown by the fact that, as already said, "*one*" is not only the name of a *number* but that of a type of twin-class *function*. And the addition of the *function* "one" to itself gives rise to a series of *functions*, which may be called two, three, four, etc., just as the addition of one to the *class* one gives rise to the other *numbers*.

Again some writers define the "product" of the two *classes* as the class made up of all pairs of things, one from each of the classes.*

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* *Fundamental Concepts*, p. 100.

It will be found that the product of two of the number-*functions* considered above agrees in name with the "product" of the two classes, just as the sum of the two functions agrees in name with the sum of the two classes. For example, the product of the *functions* "two" and "three" is "six," and again the product of the *classes* two and three gives the number six.

In brief, then, if two classes correspond to a third, their *correspondence* to each other is called a *product*, and their *correspondence* to the third a *sum*. If the correspondences happen to be *operations*, then it might perhaps be said that the result of applying two operations *successively* is a *product*, while the result of applying them *simultaneously* is a sum.

The correspondences which are multiplied to give a product are called its *factors* or components, and the *inverse* of a product is the product of the inverses of its factors.

If the correspondence of a class A to a class B is represented by

$$A = f_1 B, \text{ and, if } B = f_2 C,$$

then the correspondence of A to C, which is the product of f_2 by f_1, is represented by

$$A = f_1 f_2 C.$$

If this product is represented by a single symbol F, then we have

$$A = f_1 f_2 C = FC,$$
$$\text{or } F = f_1 f_2.$$

If the direct and inverse forms of any function f are distinguished by Df and If, then a product of direct and inverse functions is represented by such an expression as

$$Df_1 \,.\, If_2 \,.\, Df_3 \ldots, \text{ and so on.}$$

If two such products are equal, the statement to that effect may be altered in various ways, while still conveying the same truth. For example, it may be desired to have one factor as If_2 alone on one side of the *equation*, which then states that this function is equal to a certain product of factors on the other side.

Two statements, nearly self-evident, may be made use of to effect the desired *transformations* of such equations.

(1) *If two functions are equal, then the products of both by a third function are equal.*

$$e.g., \text{ if } f_1 = f_2 \text{ then } f_3 f_1 = f_3 f_2, \text{ or } f_1 f_3 = f_2 f_3.$$

(2) *The product of the direct and inverse of any correspondence is "one,"*

or
$$\mathrm{D}f_1 \,.\, \mathrm{I}f_1 = \text{one}.$$

Suppose, for example, we have

$$\mathrm{D}f_1 \,.\, \mathrm{I}f_2 \,.\, \mathrm{D}f_3 = \mathrm{D}f_4,$$

and we wish to extract $\mathrm{I}f_2$. The following steps may be employed :—

(1) Multiply both sides by $\mathrm{I}f_1$.

This gives
$$\mathrm{I}f_2 \,.\, \mathrm{D}f_3 = \mathrm{I}f_1 \,.\, \mathrm{D}f_4,$$

since the factor *one*, which is the product of $\mathrm{I}f_1$ and $\mathrm{D}f_1$ on the left side, is of course omitted.

(2) Multiply both sides into $\mathrm{I}f_3$.

This gives
$$\mathrm{I}f_2 = \mathrm{I}f_1 \,.\, \mathrm{D}f_4 \,.\, \mathrm{I}f_3.$$

Since the inverse of a product is the product of the inverses of its factors, we can obtain $\mathrm{D}f_2$, if it is desired, by simply writing

$$\mathrm{D}f_2 = \mathrm{D}f_3 \,.\, \mathrm{I}f_4 \,.\, \mathrm{D}f_1.$$

The importance of dealing with equations between *products* of functions lies in the fact that, as will be found later on, the plan adopted in this book requires such expressions as

$$+a - b + c, \text{ or } \times a \div b \times c,$$

to be looked upon as *products* of correspondences (*v.* Chap. IV.). Hence the ordinary transformations of such equations as

$$a - b + c = d - e$$

are looked upon as applications of those discussed above.

Section 14.—Illustrations of Multiplication.

The general meaning of multiplication may be illustrated by considering its meaning in the case of some of the elementary examples of correspondence already given. Taking first the correspondence of English words to French words, suppose that we have

another dictionary giving the correspondence of French words to German. Then the correspondence of English to German is the product of the correspondence tabulated in these two dictionaries.

Taking next two such correspondences as that of fathers to sons, and that of wives to husbands, the product of the first by the second would give the relationship "mother."

If we regard transitive verbs as the names of correspondences, and if the object of one verb is also the subject of another, then the correspondence of the subject of the first verb to the object of the second is the product of the two verbs.

Since vectors are correspondences of points, the interpretation of the general meaning of multiplication of correspondences requires us to look upon any one side of a triangle as representing the *product* of the vectors represented by the other two sides. It must be observed that this question of the meaning of the product or "resultant" of two vectors depends solely upon the general meaning to be given to multiplication, and we are not here concerned with any question of the compounding of velocities, accelerations, or forces. It is a matter for regret that, in such a simple example of multiplication, common usage has sanctioned the use of the word *sum* instead of product, a use which has left the word product free to be misapplied in a number of ways where vectors are concerned.

In elementary algebra the multiplication of two such correspondences as $a = \log b$ and $b = \text{cosine } c$, is represented by $a = \log \cos c$. This expression represents the true *product* of the two numerical *correspondences*, and has to be carefully distinguished from the product of the number $\cos c$ by the number $\log c$, which is represented by $\log c \times \cos c$. In the latter case we multiply *numbers*, in the former, however, we multiply *correspondences of numbers to numbers.*

Section 15.—Powers and Roots of Correspondences.

Any unique twin-class function can be multiplied by itself, the product of two correspondences being called a "*square*," the product of three a "*cube*." And the general name for such a product being "*power*."

Any power of "one" is "one," and any function multiplied by its inverse gives "one."

The square of a transposition is "one," and the cube of a treposition is "one."

A function which, when "raised to a power," gives a twin-class function is called a "*root*" of it.

Thus the "square root" of "one" is a transposition, the "cube root" a treposition.*

If we take one of the illustrations of twin-class correspondence already given, as for example "*father*," then the square of that correspondence is "grandfather," its cube "great-grandfather." Taking any operand A in a cycle f, the next operand in the order is fA, the next f squared of A, the third f cubed of A, and so on.

Therefore, as already pointed out, by "*distance*" apart of two things in an order is meant that one is some *power* (as regards the correspondence under discussion) of the other, or, if we like to put it so, that the latter is some inverse power of the former.

Any one twin-class function can be regarded as a unit of distance, or as being of unit power, and we naturally expect to find direct and inverse powers distinguished in the same way as direct and inverse directions in an order, that is, by the use of some such signs as D and I, or as $+$ and $-$.

The number of factors in a *product* of two powers is the sum of the number in each power. The notation for addition and subtraction will not be further considered here as it is dealt with in Chapter IV. It should be observed, however, that the term multiplication is always used of this combination of powers of a function, and not addition, in spite of the fact that the "indices" are added or subtracted.

We therefore meet with some further justification for the use made above of *multiplication* where vectors are concerned. For, by analogy, *the mere fact that the numerical values of two uni-dimensional vectors are added or subtracted ought not to debar us from the proper application of the word product.* And the use of the word for any two vectors must depend upon its use when the two are in the same line. We regard then the vector $+4$ as the

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* It may be observed that a transposition of only *one* cycle can have no square root, but a transposition of two cycles has. Such facts can easily be investigated by tabulating the correspondences, and remembering the meaning here attached to "multiplication." They are of importance later when some meanings of the symbols $-$ and $\div$ are to be considered.

square of $+2$, and -4 as its inverse. The product of a direct power $+4$ and its inverse -4 is the correspondence *one*, which is represented by $+$, or $-$, 0; or, if indices are used, we have, for any correspondence f, $f^0 = 1$.

As already said, it is the possibility of twin-class functions having operands in common which may be looked upon as giving rise to the consideration of *fractional distances*, such as $2\frac{1}{3}$. Since a power and a unidimensional vector are here looked upon as having essentially the same meaning, it follows that the case in question also requires us to admit fractional powers, such as $f^{\frac{1}{3}}$. For example, the "square root" of a function may be represented by $f^{\frac{1}{2}}$, for the same reasons that a "*distance*" (in the sense above used) is represented by $\frac{1}{2}$.

Section 16.—Choice of an Intermediate Class.

Not only do two classes which correspond to a third correspond to each other, but any number of classes which correspond to one class have correspondences between themselves, however they are taken in pairs.

It is convenient to select some one class as *a standard of reference, or intermediate class*, and express the correspondence of any two other classes as a product of the correspondence of one of them to the intermediate class, and of the intermediate class to the other, thus reducing the number of separate function symbols required.

In the case of correspondence by value, the intermediate class adopted is called *Money*. In the case of the correspondence of words in two languages, some international language, as *Esperanto*, provides the common medium. In the case of correspondence by simultaneity, the change of which the states form the intermediate class is called the "*Time.*"

Section 17.—Tabulation of the Correspondence of Classes to an Intermediate Class.

Suppose that the symbols of the intermediate class are written in a row, and those of the other classes in rows underneath, or that those of the intermediate class are written in a column, and those of the other classes in columns to the right of it. Now it may be

that each of these other classes is composed of the "*same*" operands, and, when that is the case, the tabulation of the INVERSES of the correspondences presents a peculiar feature of importance. It is that we may write the symbols for those operands in place of the intermediate class, and fill up the table with the symbols of the intermediate class.

At first sight this may seem a trivial detail, but its importance is seen when we consider the correspondence of functions to operands. For it is a common case then to consider the functions as related by having a class in common, and the other classes composed of the same operands in each. And it has been shown above that a table exhibiting a set of such related functions can easily be altered so as to exhibit their inverses, which, like the direct forms, correspond to the operands.

The course adopted here is to arrive at the idea of a "Group" (Chap. IV.) by gradual stages, beginning with the correspondence of operands to functions *which are not connected in any way at all.* And, in this development, the case referred to above, namely, that where the functions are related by having a quantity in common, and where the other quantities are composed of the same operands, marks an important step.

Summary of Chapter I.

It is characteristic of the modern treatment of elementary mathematics to give greater prominence than was formerly done to the idea of functionality or correspondence. In order to make this idea of correspondence the basis of mathematical teaching, it seems desirable to go outside the circle of ideas found in school algebras and geometries, and not be limited to the correspondences of numbers to numbers, and of points to points. The elementary symbolism of correspondence must then precede both ordinary arithmetic and geometry.

We can do this, however, without leaving the laboratory and all object lessons behind us, and advancing into a region of pure logic, where words, or symbols for them, appear to vie in importance with experience, and practical exercises find no place. To take such a step might be to undo what has been accomplished in recent years towards bringing mathematical teaching into closer

relations with scientific and industrial problems, and therefore the usual practical exercises must be retained, and be supplemented by others leading to the desired lines of thought.

If the new ideas are to be available in the schoolroom, the language in which they are expressed must conform so far as possible to ordinary usage, and the meanings of new words must admit of practical demonstration.

Neither in mathematical nor any other language can we escape from the necessity of leaving our fundamental words undefined, except in so far as use and experience provide an explanation. The view taken here is, that common experience is no less adequate in the case of the undefined words of modern mathematics than in the case of those formerly adopted, and that it is possible to make school practical work lend as valuable aid in the one case as in the other. Naturally teachers are more familiar with the older system of building up mathematical language, but this throws no light upon the question whether it is fundamentally easier than the more modern system.

The necessity of considering the two sides of a correspondence leads to a preliminary discussion of the meaning of "class," which in turn requires a consideration of such words as "respect" and "classification."

In order to make an elementary classification of correspondences, we are forced to consider what is meant by operands in a class, or in several classes, being "*alike.*"

Being provided with the idea of correspondence, we now consider those ideas of elementary mathematics which naturally link themselves to it, such as those of symbols, order, direct and inverse correspondence, number, multiplication and addition, distance or power, and choice of an intermediate class. A clear understanding of the meaning of multiplication of two correspondences is of fundamental importance in the method adopted in this book.

In order to give a consistent meaning to multiplication it appears necessary to depart from the common nomenclature in the case of the product of two vectors, which is usually described as their "sum."

This departure finds, at any rate some further justification when we consider the multiplication of powers of a correspondence. For, from the standpoint adopted, it appears advisable from the outset

to correlate the idea of "*distance*" and "vector" in geometry with "*power*" in algebra, so that a unidimensional vector wears the aspect of a power of some unit correspondence.

The more striking reasons for regarding such an expression as $+a+b-c$ as a product rather than a sum can, however, be dealt with only when the notation for "groups" is considered (*v.* Chap. IV.). The point to be emphasised then is that a sharp distinction must be drawn between, say, 4 and $+4$. The symbol 4 represents a number, but $+4$ a *correspondence* of numbers to numbers, and -4 its inverse, and such an expression as $+4+9-2$ represents a *product* of such correspondences, or their inverses. Similarly, in regard to the multiplication table of numbers, which forms another "group," while 4 is a number, $\times 4$ is a correspondence of numbers to numbers, and $\div 4$ its inverse.

In the ordinary treatment of arithmetic these views cannot well be introduced (although familiar to mathematicians), because no clear meaning is attached to "correspondence," to "direct and inverse," and to "multiplication" of correspondences.

Even without considering vectors, the argument in favour of explaining $+a$ as a correspondence, and $+a+b$ as a *product* of two such correspondences, seems to be a strong one.

Some readers will no doubt think that the writer is forcing an open door in laying so much stress upon the true meaning of multiplication. It must be remembered, however, that the question at issue here is the extent to which ideas of far-reaching importance can be introduced to beginners, particularly through the medium of classificatory models. Among such ideas, those which are so closely linked to that of correspondence as is the idea of multiplication naturally find a prominent place. And, as a guide to the use of symbols in elementary mathematics, stress is laid upon the methods for representing products of direct and inverse correspondences. These methods do not require the use of a notation for *numbers*, or for addition or multiplication of *numbers*, but on the contrary it is difficult to see how the latter can be understood without the former.

It is in this sense that there is much to be said for making some algebra precede arithmetic, by algebra being meant the notation for products of direct and inverse correspondences. That plan would be a preparation for what would subsequently be found,

namely, that the more important symbols of elementary mathematics occur in pairs, such as $+$ and $-$, or $\times$ and $\div$, or $\frac{d}{dx}$ and $\int dx$, and may with advantage be regarded as symbols to distinguish direct and inverse correspondences from each other (*v.* Chap. IV.).

CHAPTER II.

MULTIPLEXES.

Section 1.—Duplex Classification.

In considering the choice of an intermediate class, we have seen that any number of classes may have a one-to-one correspondence to each other. As *two* classes are called a function, so to a *set* of such classes it is convenient to give a single name, *Duplex.* We might also approach the idea of a duplex without introducing that of correspondence at all, by taking the case of a class of sub-classes where the respect D′, in which the members of the sub-class differ, is the same for each sub-class, and also the states of D′ are the same. Then D and D′ are said to form a classification and cross-classification.

The D classes, as we have seen, correspond to each other, but the D′ classes have a correspondence to each other as well. If the members of the duplex are arranged in columns and rows, then the rows correspond to each other as well as the columns.

Models of a duplex, for example, a colour-shape duplex, or a shape-size duplex, can be made in the usual way, and also there are numerous common illustrations. Thus the ordinary analysis of simple salts gives a classification of them by metal, and a cross-classification by the acid radical.

A pack of cards offers a classification by suit, and a cross-classification by value.

In a Latin grammar, nouns are classified by declension, and cross-classified by case.

Section 2.—Symbolical Representation of Things in a Duplex.

To represent members of *single classes* we can either assign a separate alphabet to each class, or use two alphabets, one to indicate class, and the other for the different members of a class (Sec. 6, Chap. I.). To represent *things in a duplex* we can either

assign a separate alphabet to each of the two respects in which the things differ; or use two alphabets, one to indicate the respect, the other to indicate the class of that respect. Finally, we can use but *one* alphabet, but make the *order* in which its symbols are written show the respect to which each refers. The last method involves arranging the respects in an arbitrary order.

Section 3.—Triplex Classification.

A triplex is made from things differing in three respects, as a duplex is made from things differing in two respects.

The idea may be approached either by considering a number of duplexes which correspond to each other, or by considering a duplex of sub-classes instead of separate individuals.

Suppose that all the duplexes which correspond have the same two respects, corresponding members agreeing in both of them. As already said, a duplex can be regarded as a correspondence of columns to each other (supposing it is arranged in columns and rows), or as one of rows to each other. Similarly, a triplex can be regarded as a correspondence of *duplexes* to each other in *three* ways. For, if we take a set of corresponding columns from the duplexes, the members of these columns form a duplex by one of the respects already considered and the third one. While if we take a set of corresponding rows, these again form a duplex by the other of the respects already considered and the third one.

The use of "*column*" and "*row*" in order to distinguish the classes by one respect in a duplex from the classes by the other respect, though convenient, is slightly misleading. For it makes no difference to the meaning of a duplex how the members are arranged, and there appears to be no third popular term for the classes by the third respect in a *triplex*. Perhaps "files" might be used.

Turning now to the second way of explaining triplexes, we have seen that, in a duplex, the respect D' of all the sub-classes is the same, and the states of it the same, for each sub-class. To arrive at a triplex, we merely suppose that the duplex is one of sub-classes, and that the members of these sub-classes all have the same values of the third respect D''. Each of the former sub-classes becomes itself a duplex.

Symbolical Representation of Things in a Triplex.

The same three methods may be used as in the case of a duplex, namely,

(1) A separate alphabet for each respect,
(2) Two alphabets, one to show respect, the other to show class,
(3) The order of the symbols may show the respect to which each refers.

Section 4.—Quadruplex Classification.

It has been seen that in a correspondence we may have on either side not a single class but a duplex, and, similarly, we now consider the correspondence of a triplex to a triplex. The members of a quadruplex differ from each other in four respects, and the idea may be approached either by considering the correspondence to each other of a set of triplexes, or by considering a triplex of sub-classes. As a triplex may be regarded as a set of corresponding duplexes in *three* different ways, so a quadruplex may be regarded as a set of corresponding triplexes in *four* different ways. Similar remarks hold of higher multiplexes.

Symbolic Representation of Things in a Multiplex.

The methods already described for a duplex and triplex can be extended to any multiplex. We can assign a separate alphabet to each respect in which things differ, or use a combination of two alphabets, or make the order in which symbols of an alphabet are written show the respect to which each refers.

Summary of Chapter II.

The ideas of Chapter II. connect themselves with those already discussed, chiefly through the fact that a number of quantities which *correspond to* each other form a duplex. The ideas of a triplex, and quadruplex, and of the higher multiplexes can easily be illustrated by models, and by common examples. In giving the latter, however, it is advisable to carefully avoid importing into the idea of a multiplex that of *order*. Thus, for example, we can indeed

arrange the members of a duplex in columns and rows, but they form no less a duplex however arranged.

Having gained the idea of a multiplex, we may proceed to use it in the further building up of mathematical ideas and language. On the one hand, we can make a mental substitution of words already explained for the operands of a multiplex. Thus, for example, we might have a multiplex of classes, or one of functions, instead of the operands. On the other hand, we can utilise the idea of things in common. Thus, for example, we can suppose that the states of a respect are also the operands of a cycle, and therefore have an order. If this is true of both the respects of a duplex, then we have an ordered duplex, or "space of two dimensions." For it will be found that, in Chapter III. which treats of Geometry, the foundations of our geometrical ideas are looked for in the theory of ordered multiplexes.

While Chapter I. is of an algebraical character, and Chapter III. of a geometrical character, Chapter II., which forms a connecting link, contains a development of the idea of classification and cross-classification, which latter might be described as one of the sources from which both algebra and geometry have sprung.

Another way of using the idea of things in common is to suppose that the operands of a multiplex are also those of a quantity in a correspondence. Thus, for example, the operands on either side of the correspondence may form a duplex, or may form a triplex. These possibilities are considered in Chapter V. which deals with Multiple Correspondence. Thus Chapters III. and V. arise, though in different ways, from the union of ideas discussed in Chapters I. and II.

The symbolical method of representing things in a multiplex will be found of importance, not only as regards the correspondence to each other of the elements of multiplexes (Chap. V.), but also as regards the correspondence of operands to functions (Chap. IV.). An important example of that kind of correspondence consists of those known as one-to-two correspondences. And in a one-to-two correspondence the problem of expressing one of the three quantities in terms of the other two will be found to be the same as that of representing a thing in a duplex. For the other two quantities play the part of the two respects of a duplex.

CHAPTER III.

SPACES.

Section 1.—Illustrations of Ordered Multiplexes.

As already said, it forms no part of the idea of a multiplex that classes of any respect should have an order. To consider ordered multiplexes demands that attention should be paid to the idea of multiplex, and to that of order, *separately first*, as has now been done, and then that the two ideas should be combined.

As illustration of an ordered duplex may be taken sets of things of two kinds, for example vessels containing apples and oranges. These can be classified by the number of things of the one kind, and cross-classified by the number of the other kind. And, since numbers form an order, the resulting classification is an ordered duplex.

Similarly, vessels containing three kinds of things form an ordered triplex, and those containing four kinds of things an ordered quadruplex.

The position of a set of things in such an ordered multiplex can be represented symbolically:

(1) By the symbols for the *number* of each kind of thing, combined with symbols to show *which kind* a number refers to;

(2) By making the order in which the *numbers* are written show the *kind* to which each number refers.

The symbolic representation of numbers has not yet been referred to, and, for the present, it may be supposed that only the symbols for the digits are known, and that, therefore, the number of things of any one kind is limited to nine. It will be found that, from the present standpoint, the ordinary notation for numbers is itself but an application of the methods of representing things in a multiplex.

Section 2.—"Pure" Geometry.

There is something to be said for calling any ordered multiplex a SPACE,* and subsequently distinguishing spaces as "continuous" and "non-continuous," and in other ways. This course will be adopted here instead of, as usually done, reserving the name space for a particular description of ordered multiplex which happens to be of greater practical importance than any other, and with which alone geometrical ideas have hitherto in school work been associated. For a "continuous" multiplex it would perhaps be convenient to reserve the term "manifold." Along with "*space*" we naturally import from the customary language of "*Geometry*" all such names as *point*, *line*, *straight line*, *dimension*, *triangle*, etc. And these words have to be given a purely classificatory meaning in the same way as has already been done for the word "position;" a process which in turn suggests axioms and deductions of a purely classificatory character.

We should have then, on the one hand, a body of undefined terms, definitions, axioms, and deductions relating to classification, and constituting "*pure*" geometry; on the other hand, the system of ordinary geometry where the same language is employed, and which now wears the aspect of *applied* mathematics. For the application to be satisfactory, certain results of experience or experiment are necessary, and, to that extent at any rate, ordinary, or workshop, geometry is certainly an experimental science. Whether we regard it as *wholly* experimental or not depends upon whether or not we regard the purely classificatory axioms such as that "quantities which correspond to the same quantity correspond

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* The term might with some reason be extended to ordered classifications which are not multiplexes, but, for the sake of simplicity, it will here be limited to multiplexes. The mere fact that the spaces considered below are non-continuous might justify us in calling their geometry non-Euclidean. Probably every geometry now taught in school is, to some extent, non-Euclidean in the sense that the meaning of the undefined terms, and the number and nature of the axioms, are rather different from those understood by Euclid.

A system of axioms and deductions in elementary geometry, which dispenses with the assumption of continuity, will be found in "Rational Geometry," by G. B. Halsted. (2nd edit. Wiley & Sons, 1907).

to each other" as based on experience, a question outside the scope of this book.

At present the result of making the step from applied geometry to "pure," or "classificatory," or "abstract," geometry is that some important subjects become clearer, but that many subjects formerly regarded as elementary become relatively difficult. For example, the idea of spaces of more dimensions than three is one which gains greatly in clearness, but that of a "continuous" curved line, such as the circumference of a circle, now presents difficulties.

Such difficulties are probably only temporary, and are due to the fact that, as the history of mathematics shows, changes in the general view of a subject can only very slowly be incorporated into teaching practice. It must be remembered that, whatever difficulties are met with, are probably at any rate logically unavoidable, since, in a sense, classificatory geometry is the only geometry we know, and that the classificatory view also offers the advantage, which appeals to some minds, of showing clearly the fundamental unity of algebra and geometry.

Section 3.—Terms used in "Pure" Geometry.

The classificatory meanings of some of the common terms of geometry may now be considered.

The classified things of an ordered multiplex become its *points*, and the respects in which the things differ its *dimensions*.

Therefore a single order of things is a space of one dimension, or *line*. An ordered duplex is a space of two dimensions, or *surface;* an ordered triplex a space of three dimensions; an ordered quadruplex a space of four dimensions. The idea of operands in common which, in the case of correspondences, leads us to the ideas of multiplication and addition, in geometry takes the form of considering spaces having points in common. Thus we may consider lines in a duplex space, surfaces in a triplex space, solids in a quadruplex space. The points of a line in a duplex space can be regarded both as a single order and as a duplex order; those of a line in a triplex space both as a single order and as a triplex order.

But besides this inclusion of spaces in one another we might devote attention to the fact that several spaces of the *same* number of dimensions may have the same points, the simplest example of

which is, that points may be arranged in an order (and so form a line by definition) in more than one way. And, similarly, the same set of points may form more than one duplex space, or more than one triplex space. It is usual, however, to greatly neglect this aspect of elementary geometry in comparison with the former one.

To describe the relative position of two points in a space we must give their relative position in each of the dimensions separately. In the simplest kind of space these positions can be expressed only by whole numbers, because, in such spaces, distances expressed by whole numbers are the only ones possible. If one point has the same position relative to another that the latter has to a third point, then the three points are said to be in a "*straight*" line.

In the case of a straight line the ratio of the relative positions of one point to another, as regards each of the dimensions, remains constant throughout its length, and defines the "*direction*" of the line, or the "*angle*" it makes with the dimensions. Any two dimensions of the space are at "*right*" angles. Two straight lines which have the same direction are "*parallel.*"

It will be seen that a proposal to begin the study of ordinary, or workshop, geometry with Cartesian Geometry would have something in common with the imaginary course under discussion. For the separate dimensions of an ordered multiplex play a part similar to that of the axes of co-ordinates in Cartesian Geometry.

That geometry might perhaps be regarded as an attempt to make in the seventeenth century the step which has only become possible in our own times. For any one-to-one correspondence can be looked upon as a selection from the duplex containing *all* possible combinations of the operands from the two quantities. And, if both the quantities have an order, this selection then becomes a "*line*" in a space of two dimensions, in accordance with the meaning here attached to "line" and to "space" and "dimension."

On account of the limited meaning attached to "space" in Descartes' time the choice of correspondences which could be considered was also limited, and moreover his investigations did not throw much light upon the question of spaces of more dimensions than three.

Section 4.—Four-dimensional Geometry.

From the standpoint adopted in this book, it will be found that the last subject cannot be looked on as very advanced, or as of merely academic interest, but, on the contrary, as one quite as vital to elementary mathematics as the meaning of multiplication, or of a line.

The real necessity for its study lies in the fact that things which differ from each other in four or more respects claim our attention as well as those which differ in three, or two, or one. And it is with differences in ordered respects or "dimensions" that the requirements of modern science demands that our geometrical ideas should be associated rather than, as formerly, merely with the application of geometry to one particular kind of difference, namely, difference of ordinary position.

Since an ordered triplex is a three-dimensional space, it is natural to enquire into the geometry of an ordered multiplex of four respects.

As a guide, we may assume that the geometry of a four-dimensional space will differ from that of a three-dimensional space in the same sort of way that the geometry of a three-dimensional space does from that of a two-dimensional space. But this suggests that the introduction of some new terms may be advisable, because it would evidently be inconvenient and misleading to have no terms for three-dimensional geometry except those necessary for two-dimensional geometry; and therefore we may conclude that it would be equally unsatisfactory to employ in four-dimensional geometry only the terms of three-dimensional geometry.

The difficulties of this geometry are chiefly due to the fact that, while a model of a quadruplex can be constructed, yet it is impossible (so far as is at present known) to make a model of an *ordered* quadruplex, that is, in the sense in which a column-row arrangement of things represents an ordered duplex, or a column-row-file arrangement represents an ordered triplex. And to this difficulty must be added the want of a satisfactory and generally accepted nomenclature, failing which, some home-made terms will be provisionally used here, to bring out the connection with three-dimensional geometry.

Corresponding to a figure in two dimensions, or a volume in

three dimensions, we will use the word "*quadlume*" for four dimensions.

As a portion of a line is enclosed or bounded by points, of a surface by lines, and of a solid by surfaces, we conclude that a quadlume is enclosed by solids. Again, because a section of a line is a point, or a point divides a line into two parts, and a section of a surface is a line, that of a solid a surface, we conclude that a section of a quadlume is a solid.

Since a line in a duplex space can be straight or bent, and a surface in a triplex space can either be flat or can be curved in various ways, therefore we conclude that solids in a four-dimensional space can differ among themselves in a similar manner. But we also conclude that these differences cannot be satisfactorily described by such words as flat and curved, because the differences between surfaces in a three-dimensional space (such as a spherical surface and a saddle surface) could not be conveniently described by using only those terms applied to lines in a duplex space. With this caution, however, it may be pointed out that, since a figure in a plane can be enclosed by a continuously curved *line*, and again a solid by a continuously curved *surface*, we therefore conclude that a quadlume can be enclosed by a "continuously curved" *solid*, or by a number of solids, which will be called its *quadface*. And we naturally expect to exist a kind of quadface corresponding to a *straight* line, or a *plane* surface, which might be called a *quane* quadface.

As the smallest number of straight lines which can enclose a figure is three, and the smallest number of planes which can enclose a solid is four, it may be expected that the smallest number of *quane* solids which can enclose a quadlume is five.

As the unit of length is enclosed by two points, the unit of area by four straight lines, the unit of volume by six plane squares, we conclude that the suitable unit of quadlume is enclosed by eight *quane cubes*. The number of unit quadlumes in one made up of such units we expect to be given by a^4, where a is the length of one edge of the quadlume.

Section 5.—Classificatory Meaning of some Common Terms.

To return to the question of the classificatory meaning of common terms, by "*motion*" of a point in a space is meant of course merely *change* of the properties concerned in the space. For example, if the space consists of bodies differing from each other in the single respect of temperature, then the only motion a point can have is an increase or diminution of temperature. If the space is a duplex one, then a point can undergo motion or displacement in either of the respects independently, or in both together.

By *extension* of a body in a dimension is meant "continuous" difference as regards that dimension. For example, a body is extended in the temperature dimension if it shows a continuous variation of temperature.

It should be noticed that one cannot detect extension in any respect apart from extension in space in the common sense as well; and also that we have refrained from discussing the meaning of "*continuous*."

By *rotation* is meant essentially the exchange of an extension in one dimension for an extension in a different dimension. For example, consider an imaginary space of points differing in temperature and hardness. And suppose that we have a portion of a straight line in the temperature dimension. If the line is rotated keeping one end fixed, that is, at the same temperature and hardness, then, in the course of the rotation, the other end approaches it in temperature, but varies from it in hardness, until, when the line has rotated through a right angle, the two points have the same temperature, but now differ in hardness.

If the rotation is continued, they begin to diverge in temperature, but to approach in hardness, until, after another right angle, they again show a difference in temperature, but an equality in hardness.

With regard to the rotation of a quadlume, it is of interest to observe that, just as a figure in a plane rotates about a point, and a solid about an axis, so a quadlume rotates about a plane.

It is not proposed to enter here any further into pure Geometry, because, in doing so, we should be departing from the immediate question of the use of models of Classifications, though by no means from the ideas of *correspondence*, of *order*, and of *multiplex*, the

elementary meaning of which is best shown by such models. A few words may, however, be said upon the most important *application* of Geometry, that to difference of common position, which, until recent years, was often regarded as being itself pure Geometry.

Section 6.—Difference of Ordinary Position.

As already remarked, when speaking of the meaning of "position" of a thing in a class, it is possible to imagine a mathematical course in which such terms as point, straight line, triangle, etc., would acquire their meaning *first* in connection with ordered multiplexes. Let us suppose for the moment that such an unusual method of teaching has been adopted, and that the beginner now turns from *pure Geometry*, the study of ordered multiplexes, to what, to him, would seem an *application* of it. His first question then is, *can* difference of common position be treated as a multiple difference, or how can one obtain an ordered multiplex of things differing in common position?

Since the parts of a body, however small, differ from each other in common position, it is clear that the "*points*" of the required space, that is, things which differ ONLY in position, can never be realised in practice, but can be represented to the eye by dots; and, similarly, that even the narrowest ink streak, or the finest thread, cannot exactly represent a "*line*" of the required space, that is, a single order of points.

Of these lines, four sorts or definitions are of great practical importance, and have the peculiarity that points which lie on a line described according to one of the four rules or definitions also lie on a line described by any of the other three rules. These lines are:

(1) The line of a crease made in paper when it is folded.
(2) The line of a ray of light.
(3) The line of a stretched string.
(4) The line of points between two fixed points of a body which do not move when the body is rotated.

If a string is stretched between the two fixed points referred to above, it coincides with the points which do not move; if a ray of light passes from one fixed point to the other, it too passes along

the same row of points; and again, if a piece of paper is folded, the points along the crease fall in the same row.

These *agreements*, the list of which might perhaps be extended by the inclusion of some other cases, ought, in elementary geometry, to be looked upon as purely experimental results, none of which can be deduced from the others.

A correspondence between points which allows us to "MEASURE" distances along "lines" in the above sense is provided by the fact that pairs of points on bodies can be marked as such that they can be brought to coincidence with a selected pair on some body. It is found experimentally that distances on different bodies which are "equal to" the distance between the selected pair of points are "equal to" each other, under all conditions so long as the temperature, and a few other circumstances, remain unaltered. Thus, we are able to mark off points along a line as being at "equal" distances apart.

Measurement of different lines between two points shows that the four kinds of lines considered above have the property of being the shortest distance between two points.

Another experimentally observed property is that two such lines need not intersect at all, but, if they do intersect, they cannot intersect at more than one point.

Measurement of lines suggests the following, among other ways, of arriving at a duplex of points:

Suppose points are taken at equal distances apart on a line of the kind described above, and that through each point passes another line of the same kind. This can be illustrated by rods attached at equal intervals, but not rigidly, to a rod which connects them. Suppose, too, that points at the same selected distance apart are taken on the cross lines beginning from the points of intersection. Now, let the cross lines be such that the points on any two adjacent lines which are *nearest* to the connecting line are themselves at the selected distance apart. In the case of the cross rods it may be supposed that they are turned about their points of intersection until the required condition is fulfilled. The following experimental fact may then be observed, namely, that not only is the first pair of corresponding points on two adjacent cross lines at the selected distance apart, but also all those which follow. However far we go along two adjacent cross lines they remain, in

the above sense, and so far as the most delicate measurements allow us to judge, at the same distance apart.

If points on the cross lines at equal distances from the first line are regarded as forming lines themselves, then it will be seen that we have arrived at a grating, or criss-cross, of lines, such that all adjacent lines are at the same distance apart, and their points of intersection form the desired duplex of points.

A second experimentally observed property of this duplex is, that any line of its points which is "straight," in accordance with the general definition of straightness of a line in a duplex (Sec. 3), is of the special kind discussed above. A circumstance which justifies the application of the purely geometrical term "straight" to the path of a ray of light, or a stretched string, etc. For example, if a straight line in the duplex is such that any point is at a distance of three units in one dimension, and two in the other from its preceding point, then the line is, so far as experiment can show, the path followed by a ray of light.

A line in any duplex of points differing in ordinary position, even if the duplex is a purely arbitrary one, is, however, straight in the mathematical sense, if it agrees with the general definition of straightness (Sec. 3), and it need not be the path of a ray of light. Again, the measurement of angles in the ordinary sense depends upon certain experimental facts (which will not be gone into here) and the practical definition of right angle, that given by Euclid, agrees with the mathematical definition adopted here only when the columns of the duplex happen to be at 90° to the rows. As already said, from a classificatory standpoint, any two dimensions of a space are at right angles, but it will be obvious that the columns and rows of such a duplex as the one described above may be at any angle to each other, as measured in degrees.

It must be remembered that the description of the experimentally observed properties of lines of the kind under discussion is one which permits of considerable variation, and it is not suggested that the outline above is superior to others which might be given. The object to be arrived at is to show the great practical importance of studying one particular kind of ordered multiplex of things differing in ordinary position; a multiplex which, historically, assisted to give rise to the language and ideas of pure geometry, but which, at the present day, should rather be the subject to

which these ideas are to be applied. So far as lending itself to purely mathematical reasoning is concerned, this particular multiplex has no virtues beyond any other ordered multiplex, and the properties which lend it practical importance should, strictly speaking, find their place in a text-book of physics rather than in one of mathematics. The part played by "geometrical instruments" in constructing a two-dimensional, or a three-dimensional lattice of points is, from a mathematical standpoint, akin to that played by scissors and paint-brush in constructing a colour-shape duplex.

Section 7.—Notation for Numbers.

As already said, an understanding of the meaning of an ordered multiplex is necessary in order to comprehend the Arabic notation for numbers, and therefore that notation should follow upon the elements of geometry.

The *simplest* methods for representing numbers are, firstly, by a number of symbols equal to the number to be represented; for example, in the Roman method three is represented by three lines; or, secondly, by giving a separate symbol to each number, as is done for the digits.

It is natural now to try to avail ourselves of the methods for representing things in a multiplex by classifying numbers as a multiplex. That is, we wish to be able to regard numbers as differing in certain respects, and to give symbols to the states of these respects instead of to the numbers themselves.

Clearly, however, no *one* multiplex will suffice, because, by assigning any finite number of symbols to the classes of the several dimensions, only a finite number of things can be represented. For example, if for the moment we suppose that some numbers have been classified as a duplex having four classes in each dimension, then the duplex contains only sixteen of these numbers, and our methods for representing things in a multiplex cannot help us beyond that.

But, IF the dimensions of a multiplex have a definite order, no *matter how many dimensions there are*, then, by adopting the method of making the order in which symbols for classes are written show the dimension to which each refers, a comparatively small number of distinct symbols can be employed to represent any

number of things however great. If some multiplex does not contain a sufficient number of things, we have only to take a multiplex with a higher number of dimensions. It will be noticed that a new idea has been introduced here, that of the dimensions of a space having a definite order, and it remains now to explain how, in the case of a multiplex of numbers, it is possible for such an order to suggest itself.

In considering illustrations of ordered multiplexes, it was pointed out that, if the members in a collection of sets of things could be classed as being of two kinds, then the collection formed a duplex, if of three kinds then a triplex, and similarly with other multiplexes. Now, instead of having sets of things, we might be dealing with sets of classes. As already pointed out, the development of mathematical ideas often takes place by this kind of substitution. Suppose that the classes are of two kinds, namely, *those containing some particular number of things and those containing a different number*. Then we can count the number of each kind of class in each set, and arrange the sets as a duplex accordingly. In any such multiplex, instead of having things of several kinds, such as apples and oranges, we have classes of several kinds, namely, such as contain different numbers. But each set contains some definite total number of things, found by multiplying each class number by its dimension number, and adding the products, and these totals therefore form a multiplex, and can be symbolically represented by one of the ways by which the positions of things in a multiplex are represented. In the Arabic notation the order in which the *class*-symbols are written is made to show to which *dimension* of the multiplex each refers, that is to say, the kind of class which has been counted.

But, since these classes differ from each other in the number of things contained in them, they form an order, so that the dimensions of the multiplex have a definite order no matter how many there are.

The particular numbers chosen for the classes characteristic of the several dimensions naturally are such as form a series of powers (those of the "radix of the scale"), because the greatest economy and simplicity of symbols is secured thereby, but, in the explanation of the Arabic system, this is a comparatively minor detail. The important thing is, to correlate that system with the general repre-

sentation of things in a multiplex, and, in particular, with the representation of vessels or collections of things of several kinds.

It may be noticed that, using "space" in the sense employed here, the idea of spaces of more dimensions than three is met with on the very threshold of mathematics, in the use of numbers of more than three digits. For all the numbers represented by two digits form together a "space" of two dimensions, those represented by three digits a "space" of three dimensions, and those represented by four digits a "space" of four dimensions.

If the order in which symbols are written is made to show the respect or dimension of the multiplex to which each refers, then the introduction of a symbol for a null or empty class becomes necessary. Without this, the case where one or more kinds of thing are absent entirely could not be represented. Thus the use of the symbol 0 is necessary in the Arabic system, but not in the Roman system.

Its use here, suggesting as it does that of a new symbol for a number, does not explain its occurrence in other ways, as in the notation for addition and multiplication of numbers, and therefore the question of its meaning has to be dealt with again later. The view of the numbers 1, 2, 3, etc., as correspondences obtained by the addition of the correspondence *one* to itself, does not allow us to include zero among numbers so defined, and therefore, from that point of view, zero cannot be termed a number.

Summary of Chapter III.

Section 1.—Two Languages of Geometry.

The union of the idea of an order with that of a multiplex is assumed to give us a "space," and to all geometrical terms a classificatory meaning can be given. Since these meanings are, at present, not those with which we first became familiar, it follows that one must consider that, in a sense, two languages of geometry exist, in which the same words are employed, but with distinct meanings.

The classificatory language is apparently that which must be employed in exact reasoning, for otherwise our arguments are not valid. But, in that language there is no place for experimental

results, for all results are contained in, or implied by, the original postulates.

The other language is that used in describing school practical work, and also perhaps in the older mathematical works, such as Euclid. It is a difficult question how far it allows of any deductions at all. For we can make experiments upon beams of light, upon the displacements of solids, and so forth, but apparently there is but one way in which we can draw conclusions from the results. Namely, we must

(1) Translate the results into classificatory language.

(2) Draw deductions in the usual way in that language.

(3) Translate back our conclusions into the language of our experiments.

And this is probably what is more or less unconsciously done in a geometrical course based on practical work.

The object of the practical exercises in such a course is, on the one hand, to show that the operands, of which we have practical experience, do fulfil the ideal conditions laid down in the postulates from which it is proposed to reason. The attainment of that object is a far from easy task.

But, in the usual school course, these practical exercises have, on the other hand, to play the part which, in this book, is assigned to classificatory models. Practical work in paper-folding, the tracing of ornamental designs, map-making, the construction of cardboard models of regular solids, and so forth, is really made the way of introducing classificatory ideas, and the language of pure mathematics is built upon these instances, instead of upon the classificatory models.

Now it seems to be true that, *historically*, such experiences and experiments, and the reasoning based on them, have led to the development of both applied and pure mathematics, but the question is, whether the usual school course is calculated to make clear the distinction between them?

The difficulties of founding *pure* mathematics upon experiments relating to ordinary difference of position are shown, amongst other ways, by the extreme slowness with which the theory of spaces of more dimensions than three was developed. Or again, we might instance the fact, which we have not hitherto referred to, that the dimensions of a space may as well be *closed* orders as open ones.

In the ideas of pure geometry the one case is as natural to consider as the other, but, in this matter, as in the previous case, practical work of the ordinary school kind does not appear capable of lending much help.

From the confusion of applied with pure geometry it results that, on the one hand, some experimental facts acquire a misleading appearance of being fundamental to pure mathematics, and, on the other hand, the amount of experimental knowledge necessary in order to make valid applications of pure mathematics to ordinary difference of position is apt to be underestimated.

It might also perhaps be argued that the habitual identification of pure geometry with its application to ordinary difference of position tends to give a greater show of probability than ought to be allowed to the suggestion that phenomena such as those of optical isomerism, and others in chemistry, depend upon configurations in the ordinary sense. The sharper distinction now drawn between pure and applied geometry should perhaps lead us to look with a certain suspicion upon some chemical theories which date from an earlier period.

Section 2.—The Part played by Experiment in Ordinary Geometry.

But, if there is an experimental science of common geometry, why, it will be asked, do we not find a progress in experiment of a character similar to that which we find in, say, chemistry or electricity?

One reason perhaps is that the confusion which reigned in geometrical language until quite recently made it difficult to even formulate a question to which an experimental answer of interest could be expected. To a person who attached a workshop, or experimental, meaning to his words a question might seem to demand an experimental answer, while, to a mathematician who attached a classificatory or logical meaning to his words, the same experiment might seem absurd, because leading to a foregone conclusion, and yet both might be right in their own ways.

Thus, even within the bounds usually assigned to practical geometry, when the part played by experiments is better understood, it seems just possible that a careful consideration of these experiments might suggest others; that the results of some of these

might be unexpected, and suggest in their turn further experiments; and that, in this way, experiments may in time give as interesting results in everyday geometry as in other branches of science. All this, however, would have no bearing at all upon the system of pure geometry, which has a basis in classificatory ideas quite apart from our everyday experience of the results of moving bodies in different ways, and of the use of geometrical instruments.

But it is when we consider the arbitrary nature of the limits which writers on practical geometry set to their subject that a more satisfactory solution appears of this difficulty as to experimental progress.

If the usual practical exercises really belong to physics, and are not pure geometry, it may be asked what distinguishes this branch of physics from others, and where is its boundary to be placed? How is it, for example, that the usual agreement of distances on bodies with each other (Sec. 6, Chap. III.) finds a place in a work on Elementary Geometry, but the usual disagreement, if the bodies happen to be brought near a fire, or near a powerful magnet does not? Is the one kind of phenomenon any less a matter of "geometry" than the other? And, if it is not, might one not say in a sense, that experimental geometry is actually making considerable progress, which, however, is masked under other names? The barrier set up between what is commonly recognised as practical geometry and what is commonly recognised as physics seems to be of an artificial rather than a natural character. An example of its breaking down at the present time is perhaps afforded by the development of the "Principle of Relativity." *

Section 3.—The Part played by Correspondence in Arithmetic and Geometry.

One end served by the investigation of respects in which things differ is, that complicated differences observed between things are resolved into differences in respects already known. Similarly, one practical end served by Arithmetic and Elementary Geometry is, to carry out certain operations or constructions on given numbers in the one case, and points and lines in the other. Both cases may

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* *v. The Theory of Relativity*, by R. D. Carmichael. (Wiley & Sons, 1913.)

be included in one by saying that the object is to effect certain correspondences. And this is accomplished by resolving them into a few simple *factors*, the *multiplication* of which give rise to the more complicated correspondences required.

In the teaching of Arithmetic and Elementary Geometry, however, this analysis of correspondences into factors is somewhat obscured by details, which, from the present standpoint, are of great, but yet secondary, importance. The chief of these are, the use of the Arabic notation in Arithmetic, and the idea of intersection in Geometry.

Since the Hindoo-Arabic notation for numbers is based upon the idea of an ordered multiplex, the view is taken here that the explanation of this notation should be closely associated with the teaching of Elementary Geometry. As a correspondence of numbers to numbers expressed in this notation must necessarily be a correspondence of multiplexes to each other, instead of a correspondence of single classes, therefore Arithmetic must be looked upon as a kind of Multiple Algebra,* and is, therefore, a subject to be considered in Chapter V.

In Arithmetic the fundamental correspondences are committed to memory in the form of the addition and multiplication tables of digits. In practical Geometry, however, certain *constructions* are relied upon to give the fundamental correspondences, instead of them being committed to memory, or obtained by reference to tables. Thus, the use of calculating machines in arithmetic would

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* The use of this name is of course not intended to imply the introduction of advanced algebra into arithmetic books. By Multiple Algebra is meant here simply the theory of the correspondence of multiplexes to each other, and both "correspondence" and "multiplex" are words which, the present writer holds, should be among the first mathematical terms with which we become acquainted. The ordinary rules for addition, multiplication, and so forth, of numbers are an illustration of the theory, and every one who knows them knows Multiple Algebra to that extent.

If the operands with which we are concerned in a correspondence can be regarded as elements of a multiplex, instead of simply elements of a class, then it may be possible to simplify the correspondence otherwise than by factorising it. It may turn out that the multiple correspondence involves but a few single one-to-one, or one-to-n, correspondences which are already known. It is this kind of simplification which we meet with first of all in Arithmetic in calculating with numbers expressed in the Arabic notation.

furnish a kind of parallel to the use of mathematical instruments in Geometry.

The two most important fundamental correspondences in elementary geometry are, the correspondence to two points of a straight line joining them, and the correspondence to a centre of a circumference having a given radius. The possibility of arriving at these correspondences is usually postulated at the beginning of a work on geometry.

While the multiplication of correspondences, for example, of vectors, is of importance in Geometry, as in Arithmetic, yet the idea of correspondences having operands in common also appears in another form in Geometry. Where the quantities with which we are concerned consist often of classes of operands, instead of single operands, it is natural to expect that their intersection will play a part, as well as the possession of an entire quantity in common. For example, the correspondence of a centre to a circumference is of a kind not met with in elementary Arithmetic, as it is one of single operands on the one side, to infinite classes on the other side. Thus we have to deal with the intersection of circles described from two centres, or, again, with the intersection of a straight line, and a circle.

Section 4.—General Relations of Algebra with Geometry.

The attitude taken up here as to the relations of algebra and geometry is, that the most important source of algebraical ideas lies in the notion of correspondences having operands in common, and the most important source of geometrical ideas lies in the notion of "spaces" having points in common. "Correspondence" and "space" are classificatory ideas which are closely connected, the idea of "classification and cross-classification" being vital to both.

Some writers have taken the view that the word space is of little importance, and scarcely demands an explanation at all, but it will be seen that it is here looked upon as of great importance to elementary mathematics, and as admitting of a simple explanation by models. Like other geometric terms, however, "space" belongs not only to pure geometry but to experimental physics, and it is the second use which presents difficulties.

It will be seen, too, that, from the present standpoint, *numbers*

play a fundamental part in elementary pure geometry. For example, the idea of relative position of two points seems to require a counting of their distance apart as regards each of the dimensions of the space in which they lie.

While numbers are essential, diagrams are not, and their use may even be misleading so far as pure geometry is concerned, although they are of course convenient in applied geometry. In elementary geometry, questions of intersections are often regarded as settled by reference to a diagram, rather than to axioms or previous propositions.

The question of the extent to which Algebra and Geometry should be TAUGHT together is scarcely within the scope of this book, which deals only with fundamental ideas. In the light thrown upon the two subjects by classificatory models, it will be seen that it is not necessary to dig deeply among their roots to discover that they have a common origin; and, therefore, it may be that the ideas belonging to this origin will prove to be the best starting place at any rate for teaching both. In most works upon the teaching of Geometry (including the well-known *Teaching of Geometry*, by D. E. Smith), such a different view is taken of Geometry and Algebra, that the question as it appears here is hardly touched upon.

CHAPTER IV.

CORRESPONDENCE OF OPERANDS TO FUNCTIONS.

Section 1.—Correspondence of Operands to Functions which are in no way connected with each other.

Since one-to-one correspondences can be illustrated by models of classifications, our plan required the consideration of those ideas, such as multiplication, which link themselves to that of correspondence. In Chapter I. the ideas of addition and multiplication of correspondences, particularly the latter, were discussed. But, in Chapters II. and III., the question of one-to-one correspondence was partly laid aside, in order to follow the development of the idea of an ordered multiplex, the object being, to show the important part played by classificatory ideas in the foundations of geometry.

In Chapter IV. we return to one-to-one correspondences, but in a new form, that of the correspondence of operands to functions. It is in this form that the beginner in arithmetic first meets with the idea of correspondence, in the shape of the addition and multiplication tables of numbers. The numerical functions or operators, $+a$ and $\times a$, which correspond to the number a in those tables, were referred to among the illustrations of one-to-one correspondence in Chapter I.

But, while scarcely suitable for illustrating the elementary idea of one-to-one correspondence, those tables also present features of difficulty which make them unsuitable as a first illustration of the correspondence of operands to functions. For example, they are infinite in extent. On this account their consideration is postponed to a later portion of this Chapter, where the important idea of a "Group" is introduced, and we begin here with simpler types of correspondence.

These correspondences, like all others, may be looked upon as a

classification and cross-classification of the things which correspond, and could be studied by means of models, though the complexity of such models would make their use disadvantageous as compared with tabulations of symbols. Only enough will be said, therefore, to point out how the study of elementary one to-one correspondences leads on to that of multiplication, addition, and other "*tables.*" For our ideas on the correspondence of a class of operands to a class of functions are naturally guided by those already suggested by the correspondence of operands to operands (Chap. I.).

On the one hand, we may ask, in reference to the class of functions, what is the simplest relationship, *if any exists*, between the functions of such a class? For it must be noticed that the bare idea of a correspondence of operands to functions does not require that the functions should be in any way connected among themselves, as, for example, by having a quantity in common.

On the other hand, we may ask how the elementary classification of functions already discussed (Chap. I.) can be applied to the correspondence of operands to such functions? For example, if the functions happen to be twin-class ones, and if the same quantity of operands is concerned in them all, then the case naturally suggests itself that the quantity of operands to which the functions correspond may itself consist of those same operands. A case which might be called a twin-class correspondence of operands to functions.

The answer to the first question is supplied by the sections above (Chap. I.) on functions having a class in common, on the choice of an "intermediate class," and on the tabulation of the direct and inverse correspondences of the other classes to the "intermediate class." We therefore begin with the supposition that the functions of the class of functions are connected by having a class of operands in common. On one side, then, of the correspondence we have only operands, on the other side functions which are connected by having a class in common.

Section 2.—Correspondence of Operands to Functions which are connected by having a Class in Common.

The operands concerned in such a class of functions form a duplex, and, from this fact, it follows that a correspondence of

operands to functions of an "intermediate class" can be viewed in at least two ways. For the rôles of intermediate class, and of the class of operands to which the functions correspond, may be viewed as interchangeable, as is readily seen by a *tabulation* of the correspondence. In tabulating the correspondence it is usual to write the symbols for the class of operands along one side of the table, and those for the "intermediate class" along the top, and it will be seen that we may consider either the correspondence of the *rows* in the table to the top row, or of the *columns* in the table to the side column, as being the class of functions. Following the course adopted in the case of the correspondence of operands to operands, it would be desirable to give common illustrations of this and the succeeding types of correspondence, such as, for example, the correspondence of stress-strain functions to different materials. But it is not so easy to find quite familiar examples as in the former case.

Section 3.—Meaning of One-to-two Correspondence, and Notation employed.

If a set of functions corresponds to operands, then the inverses of the functions correspond to them too. And, *if all the rows in the above table consist of the same set of operands*, then the inverses of the functions can be tabulated by writing the symbols for this set on the top of the table, and filling up the table with a duplex of the symbols of the intermediate class, which before were written on top (*v.* Sec. 17, Chap. I.).

The following table, in which capitals, small letters, and Greek letters are employed, represents a correspondence of the kind under discussion :—

	a	b	c	d
A	β	γ	α	δ
B	δ	α	γ	β
C	γ	δ	β	α
D	α	β	δ	γ

Fig. 1.

In such a case then, it may be considered that we have three classes or quantities, which might be denoted by three letters, as P, Q, R, to deal with, each of which may at choice be written on the side of the table, or on top, or as a duplex of symbols inside, all the rows of which are alike. And, if written as a duplex of symbols inside, then the remaining two classes play the part of the two respects of the duplex, and one of the methods already described (*v.* Chap. II.) for representing the position of a thing in a duplex, may be employed to express the class inside the table "in terms of" the other two.

For this reason, such a correspondence of operands to functions is often called a one-to-two correspondence of quantities to each other, to distinguish it from the one-to-one correspondences already considered. For here any two operands from two of the quantities define or fix an operand in the remaining quantity, whereas, in a one-to-one correspondence, a single operand in one quantity is sufficient to define an operand in the other quantity.

If, as above, the two respects of the duplex are represented by the symbols of two alphabets, then we might write $\mathrm{C}d = a$ to express the fact that, corresponding to C on the left and d on top, we have a inside. An example of this usage has already been met with in the notation $f_1 f_2 = \mathrm{F}$, for a "*multiplication table*" (*v.* remarks below) of functions.

It is another illustration of "syncopated algebra," like the notation $\mathrm{A} = f\mathrm{B}$. It must be noticed, however, that we assume that f_1 and f_2 belong to two distinct sets of symbols, just as C and d do; so that $\mathrm{C}d$ conveys exactly the same meaning as $d\mathrm{C}$, and, again, $f_1 f_2$ the same as $f_2 f_1$.

If we may call a correspondence of functions to functions a *super-function*, or *super-correspondence*, then a multiplication or addition table of functions represents a correspondence of functions to super-functions, the elementary theory of which is of course the same as that of the correspondences of operands to functions now under discussion. It will be found that a study of the notations employed for *multiplication tables of functions* is of great value in suggesting an explanation of the ordinary notation for addition and multiplication of numbers.

Section 4.—One-to-n Correspondences.

Putting aside for the moment the correspondence of functions to super-functions, the order in which correspondences have already been considered above, namely, first the correspondence of operands to operands (Chap. I.), and then that of operands to functions, suggests taking as the next subject the *correspondence of operands to one-to-two functions*, and, after that, the correspondence of operands to the last kind of correspondence.

If suppositions similar to those made in the case of one-to-two correspondence are made for the correspondence of operands to one-to-two functions, then it will be seen that we may consider ourselves as dealing with four classes, any one of which may be expressed as members of a triplex in terms of the other three. This fact leads such correspondences to be called one-to-three correspondences, and similar facts lead the subsequent correspondences to be called one-to-four, one-to-five, and so forth, correspondences.

The fact that such a correspondence exists between some quantities, A, B, C, etc., may be expressed in several ways, apart from those suggested by the ways of representing the members of a multiplex. Thus, we may write $A = f\,(B, C, \text{etc.})$, which recalls the notation $A = fB$, used for one-to-one correspondence.

Section 5.—Classification of One-to-n Correspondences.

Turning now to the second question suggested by one-to-one correspondence (*v.* Sec. 1 above), namely, the classification of correspondences of any one of the above types, it will be seen that the following possibilities, amongst others, present themselves, because similar possibilities occurred in the case of one-to-one correspondence:—

(1) We may be dealing with the one-to-two, and so forth, correspondence of *classes*, instead of individual operands.

(2) One or more of the quantities concerned may have some of its members "*alike*," instead of all different.

(3) The "*same*" members which appear in one quantity may appear in one or more of the others. In short, the idea of a "twin-class" correspondence is of as much importance here as in the case of one-to-one correspondence.

(4) The quantities involved may themselves be multiplexes instead of single classes. This possibility, however, we have not yet taken into account for one-to-one correspondence. "Multiple" one-to-one, and one-to-n, correspondences are considered in Chapter V.

(5) The members of a quantity may be functions, of any type, instead of operands. As already said, a multiplication table of functions presents an example of this case.

Needless to say, these cases are far from exhausting all the means of classifying correspondences of any one of the types of correspondence under discussion. The possibilities (1) and (5) above are suggested in the manner of which we have already had numerous examples, namely, by the process of substituting various classificatory terms for the operands.

Section 6.—The Case where all the Quantities consist of the same Operands.

Case (3) is the most important, and the following table presents an illustration :—

	a	b	c	d
a	b	a	d	c
b	c	b	a	d
c	d	c	b	a
d	a	d	c	b

Fig. 2.

It will be seen that here the three quantities, which will be called A, B, and C, consist of the "same" set of operands, a, b, c, d,* and the table may therefore be compared with the twin-class correspondences discussed in Chapter I., with which were associated the ideas of "*order*" and of "raising to a power."

It will be found that several ideas which are connected with that of multiplication of one-to-one correspondences and which

* In the language adopted in "*Fundamental Concepts*" (p. 88), the table represents an "*operation*" upon the members of the set a, b, c, d; and the "*results*" belong to the same set.

either were, or might have been, introduced in Chapter I., also find an application in regard to one-to-two correspondence.

It will assist in making clear the nature of these applications if we begin by considering tables of functions which present the same features as the above table of operands. Such a table of functions is provided by a multiplication table where the "*same*" functions occur in the three quantities of functions. In other words, the product of any two functions taken from the set of functions is some function belonging to the set.

The following table may be supposed to represent such a multiplication table of functions, f_1, f_2, f_3, f_4. It shows, for example, that the function f_1 is the product of f_2 and f_3, or that f_1 "divided by" f_2 gives f_3, or that f_1 "divided by" f_3 gives f_2:—

	f_1	f_2	f_3	f_4
f_1	f_2	f_3	f_4	f_1
f_2	f_3	f_4	f_1	f_2
f_3	f_4	f_1	f_2	f_3
f_4	f_1	f_2	f_3	f_4

Fig. 3.

This table represents a correspondence of super-functions to functions, instead of one of functions to operands, like the previous table. It may be noticed that among these super-functions in the table above occurs the super-function "one," which is shown as corresponding to f_4.

Section 7.—Such Correspondences may or may not be "Permutative."

The question naturally suggests itself, does the same super-function which corresponds to a function, as f_3 at the side, also correspond to f_3 when we take f_3 on top? Or, turning to the previous table, does the same function which corresponds to an operand, such as b at the side, also correspond to b at the top? These questions can be answered by actually tabulating the two super-functions in the one case, or the two functions in the other case. It will then be seen that the question can be answered in

the affirmative for the table of functions, and in the negative for the table of operands.

We have here, then, a distinct property of such tables, which is of interest in regard to the notation for expressing an operand inside the table in terms of those on top and side. Suppose that the order in which the symbols for operands on top and side are written is made to show to which set a symbol refers. In the one case it is a matter of indifference whether we write, say, $f_1 f_2$ or $f_2 f_1$ for a function inside the table, while in the other case bc has a different meaning to cb. If the order is a matter of indifference, the table is said to be *permutative.*

This word could also have been introduced in Chapter I., merely in regard to the multiplication of two one-to-one correspondences. Two such functions are said to be relatively permutative if f_2 applied to f_1 gives the same product as f_1 applied to f_2. Thus the word is of use either when *two* correspondences of operands to operands are in question, or when a *single* correspondence of operands to function is in question.

Section 8.—The Permutative and Non-Permutative Forms of a Table.

To a function in a multiplication table of functions corresponds not only the direct but the inverse of the super-function, and, similarly, to an operand in a table of operands, such as that above, not only corresponds a direct, but an inverse, function. Now, if the term "*direct*" is arbitrarily assigned to the functions of the multiplication table as actually tabulated, then it is natural to also apply it to any super-function which corresponds to a function as multiplier.

That is to say, the correspondence of the top row to a row inside, or of the side column to a column inside is called "direct," while their opposites are called "inverse." Thus, to multiply by any function of the table is, by this convention, to apply a "direct" super-function, and to divide by a function is to apply an "inverse" super-function.

As already said, if the table is permutative, then the same direct and inverse super-functions which correspond to a function at the side also correspond to it on the top.

Now, supposing that the table is rearranged so as to tabulate

the inverses of the super-functions corresponding to, say, the side column, instead of the direct forms, is it still a permutative table? It will be clear that, in general, it is no longer so. For, suppose we take the top row and side column of a table, and fill up the inside *so as to make it permutative*, this condition limits to a considerable extent our freedom of choice in filling up the places, as a trial will show. For to every row there has to appear a column having the same symbols. And if now it is proposed to make the table which shows the inverses of the super-functions also permutative, it will be found that, except in very simple cases, this condition is in conflict with the first; so that, though it is possible to invent a table to satisfy one condition, it is not generally possible for it to satisfy both. Thus, if a multiplication table is permutative, we must expect to find that the corresponding division table is non-permutative.

If a table is not permutative, then it is no longer a matter of indifference in what order we write the symbols for two operands from top or side, supposing we use the method of writing them in order.

Instead of that method, we could combine with the symbols for the operands two other symbols, which would serve the same end as writing the operand symbols in a fixed order, that is, they would distinguish an operand at the side from one on top.

This is merely an application of the general methods for representing things in a multiplex, which have already been explained in Chapter II. For, as already said, the operands inside the table form a duplex, and the top row and side column may be regarded as playing the part of the two respects of the duplex.

If we are concerned with the permutative and non-permutative form of a table, it is clear that we are almost forced to adopt the method last mentioned for distinguishing the two respects of the duplex in the case of the non-permutative form. For, if we merely wrote a pair of symbols in a definite order it would probably be uncertain which form of the table they referred to.

Section 9.—Notations used for a Division Table.

In the case of a division table of functions, the two symbols necessary to distinguish the two respects of the duplex *are already*

provided by the two used to distinguish "direct" functions from "inverse," such as the letters D and I, or the symbols $\times$ and $\div$, or $+$ and $-$. For, to divide by a function is the same as multiplying by its inverse, or, in accordance with the convention already explained, it is the same as applying an inverse super-function. Thus, if the non-permutative form of a multiplication table is tabulated, the top row may be regarded as consisting of direct functions and the side column of their inverses, or else the side column of direct functions and the top of inverses.

Although the table is non-permutative, yet it stands in a simple relation to a permutative one, because the inverse of a product is the product of the inverses of its factors.

Thus, if a function inside the table is represented by, say, the product $Df_1\ If_2$, then $Df_2\ If_1$, which, if the table were permutative, would represent the same function, does represent its inverse.

Not only can the same two symbols used to distinguish direct and inverse functions in the multiplication table be used to distinguish the direct and inverse super-functions corresponding to them, but, if the function *one* occurs in the table, it is clear that the super-function *one* must correspond to it, and may be represented by the same symbol. In the table represented above, Fig. 3, the super-function *one* corresponds to f_4, which must therefore be a symbol for the function one, since we have supposed that we are dealing with a multiplication table.

Section 10.—A Multiplication Table of Functions is also a Multiplication Table of Super-Functions.

The table represents a multiplication table of the super-function as well as of the functions. That is to say, the super-function corresponding to the product of two functions is itself the product of the super-functions corresponding to the two factors. In symbols, if $F = f_1 f_2$, then the correspondence of functions to functions obtained by multiplying a set of functions by F is the product of the correspondences obtained by multiplying the members of the same set by f_2 and by f_1.

The importance of this observation lies in the fact that, as already said, all the tables under discussion may be regarded as correspondences of functions to operands, or of super-functions to

functions. Thus, since a *multiplication table* of functions, such as the one above, is also a *multiplication table* of the super-functions, the question naturally suggests itself, under what circumstances is a table of operands *also a multiplication table of the functions corresponding to them?*

For example, in the ordinary addition table of numbers we have a correspondence of numerical functions to numbers. These numerical functions were referred to as an illustration of correspondence in Chapter I. Can, then, the addition table of numbers be regarded as a *multiplication table* of these numerical correspondences? It will be found that this question can be answered in the affirmative, and that the result throws light upon the notation employed for addition and subtraction and the symbol for zero, because it allows us to correlate this notation with that for multiplication, division, and the symbol for *one.*

Section 11.—The Multiplication Table of Numbers.

As already said, the numbers 1, 2, 3, 4, etc., may be regarded as correspondences obtained by the addition of the correspondence *one* to itself; a view possessing the advantage that a study of the multiplication of such correspondences and of their inverses naturally suggests itself, and that the remarks already made on multiplication tables of functions apply also to this one. An examination of the ordinary multiplication table of numbers (*v.* Fig. 4) shows that it is permutative, and that it can be put in a non-permutative form (*v.* Fig. 5) by tabulating the inverses of the numerical functions which the table shows to correspond to the numbers.

Thus, we naturally expect to find two distinct symbols attached to numbers to show their quotient, or, if not, that in some way the order in which the symbols for the numbers are written will be made to show which is numerator and which denominator. Actually both methods are in use, the two symbols $\times$ and $\div$ being employed, and also the plan of writing the numerator above the denominator.

In accordance with what has been said, the multiplication table is also a multiplication table of direct numerical correspondences distinguished by the sign $\times$, and their inverses distinguished by $\div$.

Thus, while 4 represents a number, $\times 4$ represents a correspondence of numbers to numbers, and $\div 4$ its inverse. The product of the correspondence $\times 4$ by its inverse is, as in all cases, the correspondence *one*, the direct and inverse of which are of course the same, so that we may write indifferently $\times 1$ or $\div 1$.

Multiplication Table of Numbers.

	1	2	3	4	5	6	7	8	–	–	–
1	1	2	3	4	5	6	7	8	–	–	–
2	2	4	6	8	10	12	14	16	–		
3	3	6	9	12	15	18	21	24			
4	4	8	12	16	20	24	28	32			
5	5	10	15	20	25	30	35	40			
6	6	12	18	24	30	36	42	48			
7	7	14	21	28	35	42	49	56			
8	8	16	24	32	40	48	56	64			

Fig. 4.

Non-Permutative Form of Multiplication Table of Numbers.

	1	2	3	4	5	6	7	8	9	10	11	
1	1	2	3	4	5	6	7	8	–	–		
2	–	1	–	2	–	3	–	4	–	5	–	6
3	–	–	1	–	–	2	–	–	3	–		4
4	–	–	–	1	–	–	–	2	–	–	–	
5	–	–	–	–	1	–	–	–	–	2	–	
6	–	–	–	–	–	1	–	–	–	–	–	
7												
8												

Fig. 5.

The multiplication table of numbers presents a feature of difficulty, absent from those already considered, in the facts that it is infinite in extent, and that the numerical correspondences are not "twin-class" ones in the strict sense in which that term was used (Chapter I.), because not every operand which appears in the top row appears also inside the table. The numbers appearing in any row inside the table are a *part* of the infinite class forming the top row.

From this it follows that when the non-permutative form is tabulated (*v.* Fig. 5) gaps appear inside the table which can be filled only by inserting the product of a direct and inverse number. For example, there will be no single number for the product $\times 5 \div 2$ or $\frac{5}{2}$, because 5 does not appear in the row corresponding to 2 at the left-hand side in Fig. 4.

The methods by which equations between products of direct and inverse functions (*e.g.*, $\mathrm{D}f_1 . \mathrm{I}f_2 . \mathrm{D}f_3$) can be altered, have already been referred to in Chapter I., and can be applied to equations between such products as $\times 5 \div 2$.

On the one hand, such a product can be altered in appearance by multiplying the direct factor by some factor P, and the inverse factor by the inverse of P $\left(\frac{a}{b} = \frac{ma}{mb}\right)$, since P multiplied by its inverse is one.

On the other hand, if a direct factor and its inverse already occur in the product, these factors may be omitted ("cancelling like factors in numerator and denominator") without affecting the value of the product.

Again, a factor as $\times 5$ on one side of an equation can be transferred to the other by multiplying both sides by the inverse of that factor. The factor then disappears on its original side of the equation, because the product of direct and inverse is *one*, but reappears on the other side in its inverse form.

A consideration of the relations between the multiplication and addition tables of numbers will be postponed until the latter table has been dealt with, and it has been shown that, like the multiplication table, it can be regarded as a multiplication table of numerical correspondences.

Section 12.—Use of the Word "Substitution."

For this purpose it is necessary to refer to a property of tables of the kind under discussion, known as the associative property. It is one which, like the permutative property, can be connected with the multiplication of one-to-one correspondences. It requires, in fact, an extension to one-to-two correspondences of the idea of multiplication itself.

The question which may be asked is: how the meaning already given in Chapter I. to multiplication of one-to-one correspondences can be extended to tables like those discussed above?

It will be remembered that, in Chapter I., the meaning of multiplication of two correspondences was got by considering the case where they have a quantity in common. Then the correspondence of the remaining two quantities to each other was called a "*product*" of the two first, or factor, correspondences. Now, clearly two one-to-n correspondences may likewise possess a quantity in common, and, in that case, the remaining quantities have a correspondence to each other. A familiar example is furnished by the addition and multiplication tables of numbers, which are two distinct tables in which the same set of operands appears. It will be seen that, in a sense, one may look upon a sum of products $(ab+cd)$ or a product of sums $((a+b)\times(c+d))$ as a multiplication of the one table by the other.

But, though the idea of multiplication is as valuable here as in the case of one-to-one correspondences, the use of the *word* product is open to some objection. If we merely speak of *the product* of two one-to-n correspondences, there may be room for uncertainty which of the various quantities in the two functions is the one possessed in common. Again, a one-to-one correspondence has but two quantities which can enter into other functions giving rise to products, while a one-to-two correspondence has three, a one-to-three has four, and so on. Thus, if more than two factors tables are concerned, a number of new correspondences suggest themselves, all alike deserving to be called "products" of the original tables.

Now, if we consider the symbolic multiplication of two correspondences $A=f_1B$ and $B=f_2C$, it will be noticed that f_2C takes the place of B in the expression $A=f_1B$. Similarly, if $A=f_1(B, C, D)$ and $D=f_2(E, F)$ then we might substitute $f_2(E, F)$ for D and write $A=f_1(B, C, f_2(E, F))$, which represents a product of f_2 by f_1.

Thus, in dealing with one-to-n correspondences, the word SUBSTITUTION plays as important a part as *product* does in the case of one-to-one correspondence, its use being due to the need for avoiding language which might convey more than one mean-

ing.* *A substitution is, however, essentially a case of multiplication*, when employed as above, and it will be found that those ideas which are of importance in regard to the multiplication of one-to-one correspondences, such as the *permutative property*, or the *idea of raising a twin-class function to a power*, again appear when we consider the multiplication of one-to-n correspondences.

Section 13.—The Associative Property.

Thus, for example, a one-to-two correspondence is "raised to a power" by substituting for an operand, as b, in $a=f(b, c)$, two operands which the table shows to correspond to it, substituting again for one or both of these, and continuing the process as often as desired. This is always possible provided that any operand which appears on top or side appears also inside the table. If the table is non-permutative, substitution for an operand on top must be distinguished from substitution for an operand at the side, but, if the table is permutative, it is a matter of indifference. We thus arrive at a set of operands from which, by reversing the above process, namely, by *combining* again the operands two at a time, we can pass back to the original expression $a=f(b, c)$. This re-combining of the operands may be compared to taking a root of a one-to-one correspondence.

An obvious question suggests itself here, namely, do all ways of combining the operands lead back to the same origin? If this be the case, then the table is called "*associative*," because combining or "associating" the operands in different ways does not affect the result. To use the notation adopted in the "*Fundamental Concepts*" (p. 90), an operation o is associative if $ao\,(boc)=(aob)\,oc$.

The drawing up of some imaginary tables will show that a table may be associative, but not permutative, or permutative, but not associative.

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* Another reason is that it is more difficult to separate the symbols for one-to-n *correspondences* from those for variables, than in the case of one-to-one correspondences. It is not unusual even in elementary work to use such symbols as f_4, f_2, log, tan, instead of f_1x, f_2x, log x, and so on, and to write a product as f_1f_2, but such a correspondence as $z=(x+y)^2$ is seldom represented in elementary algebra, except in the way above, which *requires the appearance of the letters x and y.*

If the functions which correspond to the operands are made to be the powers of some one cycle, then it will be noticed that the resulting table is both associative and permutative.

A test of the ordinary addition table of numbers will show that it is associative and permutative, and it may be observed that the numerical correspondences appearing in it might be regarded as powers of a single infinite cycle.

From the meaning of multiplication of correspondences, it is clear that any multiplication table of functions must be associative.

Section 14.—The Meaning of a Group.

By a "Group" is meant a special kind of correspondence of operands to functions, or of functions to super-functions, and, in its definition, the associative property plays the chief part. The term was originally "Group of operations," that is, a set of operations, forming a group or class by the possession of certain properties. From the present standpoint, the operations of such a group form a correspondence of functions to super-functions, and the associative property is the most important characteristic of the group.

Not only is a multiplication table of functions associative, but also *an associative table of any operands is a multiplication table of the functions corresponding to them.* For if boc represents the operand inside the table corresponding to the operands b and c on side and top, then it is also the result of applying to c the "direct" function belonging to b. And $ao\,(boc)$ is the result of applying to the operand (boc) the function belonging to a, so that $a\ o\ (boc)$ is the result of applying to c the *product* of the functions. But, since the table is associative, $a\ o\ (boc) = (aob)\ oc$, that is to say, the function belonging to the operand (aob) must be the product of those belonging to the operands b and a.

If the table is a finite twin-class one, then, to any operand as b at the side, there will also be that operand somewhere in the corresponding row inside. Hence, there will be on top an operand c such that $boc = b$. But, from the fact that the table is associative, it follows that there is only one operand with this property. For, if we take an operand other than b, as a, then $ao\,(boc) = (aob)\ oc$. But $(boc) = b$, therefore $ao\,(boc) = aob = d$, say. Therefore $doc = d$.

But we can arrive at all the operands of the table in turn by combining different operands with b. Therefore, whatever operand x is arrived at,

$$xoc = x.$$

Thus, the function corresponding to the operand c must be the function *one*. Thus, the correspondence *one* must occur among those which belong to operands in any twin-class *associative* table.

For the same reason, viz., that we suppose the table to be a twin-class one, the operand c must itself occur in the row corresponding to b. Hence, there must exist on top of the table an operand y such that

$$boy = c.*$$

So far as finite tables are concerned, it appears that a "Group" might be defined as a twin-class table which is associative, or perhaps merely as an associative table.

Section 15.—The Addition Table of Numbers.

The addition table of numbers (shown in Fig. 6) is, as already said, associative, and represents a multiplication table of the numerical correspondences belonging to the numbers, or of the inverses of those correspondences. The non-permutative form, or subtraction table (*v.* Fig. 7), shows products of direct forms by inverse forms.

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* If the table is infinite, the last two results cannot be deduced merely from the associative property, because then we cannot say that either b or c occur inside the table at all. Thus it is usual to postulate them in defining an infinite "Group" (*v. Fundamental Concepts*, p. 89). This makes, however, the abstract definition of a Group difficult to follow as compared to the simpler explanation which is sufficient for a finite twin-class table.

It also carries the disadvantage that, if the addition table of numbers is regarded as a group, then the postulates require us to assume the existence of a number o; while the introduction of o in the multiplication table requires us to deny that the postulates hold for the latter table, so far as o is concerned. It will be found that the position adopted in this book is, that there is no need to include o among numbers, and that, therefore, it need not appear in either the addition or multiplication tables. It is only the symbol $\pm o$ which is unavoidable.

Addition Table of Numbers.

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	–
2	3	4	5	6	7	8	9	10	–	–
3	4	5	6	7	8	9	10	11	–	
4	5	6	7	8	9	10	11	–		
5	6	7	8	9	10	11	12	–		
6	7	8	9	10	11	12	–			
7										
8										

Fig. 6.

Non-Permutative Form of Addition Table.

	1	2	3	4	5	6	7	8	9	10	–	–
1	–	1	2	3	4	5	6	7	8	–	–	
2	–	–	1	2	3	4	5	6	7	–	–	
3	–	–	–	1	2	3	4	5	6	7	–	
4	–	–	–	–	1	2	3	4	5	6	–	
5	–	–	–	–	–	1	2	3	4	5	–	
6	–	–	–	–	–	–	1	2	3	4	–	
7												
8												
9												

Fig. 7.

From the present standpoint, since the numbers are obtained by the addition of *one* to itself, there is no number which, added to another, produces no change in that other. Thus, looking only at the numbers, gaps must appear in the subtraction table, corresponding to the appearance of the same numbers on top and side. But, regarding the table as a multiplication one, these gaps can be filled at once by a symbol for *one*, since the product of a direct function by its inverse is one. Thus it is the symbol $+o$, or $-o$, which has a perfectly clear meaning, and plays the same part for the addition table as $\times 1$, or $\div 1$, for the multiplication table. The question whether we are bound to invent a symbol o for the numerical function "one" to *correspond to*, and regard o as a *new*

number, is a different one, and, in this book, is decided in the negative (*v.* Summary of Chap. IV.).

From what has been said, it will be seen that the distinction of "positive" and "negative" does not, strictly speaking, apply to numbers at all, but only to the numerical correspondences belonging to them in the addition table. The product of a "positive" operation, and its inverse, or "negative" is "*one*," for which the symbol ± 0 is adopted.

Such an expression as $+a-b+c$ is then to be regarded as a product of direct and inverse numerical correspondences; or as a product of direct and inverse unidimensional vectors, if by $+a$ is meant a correspondence of points a units apart.

As with any product, it can be altered by the removal or insertion of pairs of direct and inverse factors, and a factor occurring on one side of an equation can be removed, and its inverse written on the other side. The following statements on the left should be compared with those on the right:—

(1) $\dfrac{a}{b}=\dfrac{ma}{mb}$ $\qquad a-b=(a+c)-(b+c)$

(2) $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}$ $\qquad (a-b)+(c-d)=(a+c)-(b+d)$

(3) $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{ad}{bc}$ $\qquad (a-b)-(c-d)=(a+d)-(b+c).$

A difference from the multiplication table lies in the fact that, if p is greater than q, then $p-q$ is always supplied by the table, while in the multiplication table gaps may occur for such products. Since the inverse of a product is the product of the inverses of its factors, $q-p$ is simply the inverse of $p-q$.

Section 16.—The "Distributive" Property.

The extension of the idea of multiplication from one-to-one to one-to-n correspondences naturally raises the question whether the "multiplication" of two tables, such as the multiplication and addition tables of numbers is, or is not, *permutative?*

If $\quad A = f_1(B, C) \qquad C = f_2(D, E)$

then $\quad A = f_1(B, f_2(D, E))$,

and we have to consider what meaning can be attached to the statement that f_1 and f_2 are "permutable," or that "it is a matter of indifference which is applied first" to the quantities concerned, namely, B, D, E.

It will be seen that the same difficulties arise as were referred to in regard to the use of "product" itself where tables are concerned. A number of interpretations suggest themselves, for example, that

$$f_1(Bf_2(DE)) = f_2(B, f_1(DE))$$

or, again, that $\quad f_1(Bf_2(DE)) = f_2(f_1(BD), f_1(BE))$.

If the latter statement is true for f_1 and f_2, then f_1 is said to be "distributive" as regards f_2. For example, the multiplication table of numbers is distributive as regards the addition table, since, in the common notation $a(b+c) = ab + ac$, a notation which explains the origin of the term distributive. But the addition table is not distributive as regards the multiplication table, because $a + bc$ is not the same as $(a+b)(a+c)$.

It will be seen that, although the idea of two factors being *permutative* does not allow of a single straightforward interpretation where one-to-n correspondences are concerned, yet the idea of "distribution" may be regarded as a kind of special case of it.

The fact that the multiplication table of numbers is distributive as regards the addition table allows us to enlarge the latter by the inclusion of fractions, such as $\frac{2}{5}$. For

(1) Any two fractions, as $\frac{2}{5}$ and $\frac{1}{3}$, can be brought to the same denominator by the application of the principle that

$$\frac{a}{b} = \frac{ma}{mb}. \quad \text{Thus, we get } \frac{2 \times 3}{5 \times 3} \text{ and } \frac{5}{5 \times 3}.$$

(2) The sum can be written as

$$\frac{1}{15}(2 \times 3 + 5),$$

because multiplication is distributive to addition.

Section 17.—The "Inversor" is distributive over a Product.

Any one-to one function f_1 is distributive as regards a one-to-n correspondence $f_2(b, c, d)$, when

$$f_1 f_2(b, c, d) = f_2(f_1 b, f_1 c, f_1 d).$$

It will be seen, therefore, that the distributive property of the multiplication table might be expressed by saying that a super-function, as $\times 3$, is distributive over a product, as

$$(+2+3-5).$$

Thus the question suggests itself, can any general principle be laid down as to the super-functions which are distributive over a *product* of direct and inverse functions?

It will be found that the two kinds of super-functions which most readily suggest themselves possess this property, and that the use of a *positive or negative number as a multiplier* of positive or negative numbers, *e.g.*, of -2 in $(-2) \times (3+5-1)$, is really an example of the use of both of them.

One of these super-functions is the correspondence of a direct function to its inverse, for example, of $+4$ to -4, or of $\times 6$ to $\div 6$, and, in general, of f^{+1} to f^{-1}. So far as any one pair of signs, those for direct and inverse, is concerned, this super-function is evidently a *transposition* of a single cycle. As it is a transposition, its square must be *one*, and, since it consists of but one cycle, it can have no square root.

It is convenient to have a general name, such as "*Inversor,*" for this super-function. Since the inverse of a product is the product of the inverses of its factors, it is clear that an inversor must be distributive over a product. For example, if for the moment we denote the inversor by I, then

$$\mathrm{I}(+a-b+c) = -a+b-c$$

or, again, $$\mathrm{I}(\times a \div b \times c) = \div a \times b \div c.$$

For an "inversor" and "one" it is convenient to use the *same two* symbols as are employed to distinguish direct from inverse, though it is purely a matter of convention which of them is selected for the inversor. For example, the correspondence of

$$+ \text{ to } -$$

and $$- \text{ to } +$$

is usually represented by the sign $-$, while the correspondence of

$$\times \text{ to } \div$$

and

$$\div \text{ to } \times$$

is usually represented by $\div$.

Therefore, with the above usage, we have

$$\begin{aligned} &-(+a-b+c) \\ =& -a+b-c \end{aligned}$$

and

$$\begin{aligned} &\div(\times a \div b \times c) \\ =& \div a \times b \div c. \end{aligned}$$

It is important in regard to these "laws of signs" to distinguish between what is conventional, or a mere matter of convenience, and what is a necessity.

(1) It is a *convenience* to use the same two symbols which distinguish direct and inverse, to mean the inversor and one.

(2) It is a matter of convention that $+$ is chosen for "one," and $-$ for "inversor." The opposite plan could have been adopted, and similarly with regard to $\times$ and $\div$.

(3) If, however, $+$ is used for "one," and $-$ for "inversor," then the "law of signs" (*e.g.*, that $- \times - = +$) is merely a statement of the fact that the square of a transposition is one.

Thus the use of the term *law* in "law of signs" is neither entirely justifiable, nor entirely wrong, since, up to a certain point, we are governed by convenience alone, while beyond it we have to pay attention to the fundamental meaning of multiplication of correspondences.

It is usual in school books to somewhat exaggerate the element of necessity at the expense of that of convention; for example, the possible use of $+$ as inversor is not considered at all. With that use we should have

$$a+(b+c)=a-b-c$$
$$a-(b+c)=a+b+c.$$
$$a-d(b+c)=a+db+dc$$
$$\frac{-a}{-b}=\frac{-a}{b}, \qquad \frac{a}{b}=\frac{-a}{b}$$
$$(a-b)\,(a-b)=-a^2+2ab-b^2$$

and so on.

Again, since a transposition of one cycle has no square root, $\sqrt{+a}$ would be "imaginary," just as $\sqrt{-a}$ is "imaginary" when $-$ is taken as inversor.

It will be shown later that, though in a sense a meaning can be given to $\sqrt{-a}$, yet to do so we have to depart somewhat from the meaning hitherto attributed to the inversor. Strictly speaking, $\sqrt{-a}$ is as "impossible," or "imaginary," now as when it was first introduced, and must always be so, since a transposition of only *one* cycle can have no square root.

Section 18.—A "Tensor" is distributive over a Product.

The second super-function referred to above as being distributive over the factors of a product, is the correspondence of a function to some particular power of it, for example, that of f_1 to f_1^3. To take this case, the function corresponding to $+2$ would be $+6$ (since $+6$ is the cube of $+2$), and the function corresponding to $\times 2$ would be $\times 8$. The term "*Tensor*" might conveniently be used as a general name for this kind of super-function, since a tensor applied to a vector stretches it out; for example, the tensor above changes the vector $+2$ to $+6$. From the meaning of multiplication it will be clear that a Tensor is distributive over a product, provided the factors are permutative.

The distributive property of the multiplication table, as regards the addition table of numbers, may be looked upon as an example of this property of a Tensor. For to multiply $+2$ by 3 is really to cube the numerical correspondence $+2$. Thus,

$$3(+2+5-4) = +3{\cdot}2+3{\cdot}5-3{\cdot}4,$$

just as, to take another example,

$$\left(\frac{2\times 5}{4}\right)^3 = \frac{2^3\times 5^3}{4^3}.$$

The multiplication table of numbers might therefore be regarded as a multiplication table of Tensors, using "multiplication" in the strict sense in which it is applied to correspondences. Owing to the facts that the multiplication is permutative, and that the same numbers appear in the addition table, we meet with the peculiarity

that a tensor, as the *cubing* tensor, applied to the correspondence $+2$, is the same as applying the *squaring* tensor to $+3$.

If a positive or negative number is used as a multiplier, this is merely to apply a compound operator, consisting of a tensor and either the Inversor or one. The operator as a whole is distributive, since its two factors are distributive.

It has already been pointed out that there seems to be no need to consider the inclusion of the symbol 0 in the multiplication table of numbers, although ± 0 is a necessary feature in the notation for addition and subtraction. But, since $f^{+a} \times f^{-a} =$ "one," it follows that ± 0 may be regarded as a peculiar kind of Tensor, namely, the operator which converts a twin-class function into "*one*." Therefore, since, as shown above, the numbers of the multiplication table play the part of Tensors to the numerical correspondences $+2$, $+3$, etc., we can also consider the application of this operator ± 0 to those correspondences. Its effect is to change them to "*one*," that is, to ± 0: *e.g.*, ± 0 applied to $+3$ gives ± 0, and applied to ± 0 gives ± 0. The *inverse* of this operator, that is $\frac{1}{0}$, is indeterminate, or vague, when applied to "*one*," because the inverse of such a correspondence as

$$\begin{array}{cc} a & b \\ c & b \\ d & b \\ \multicolumn{2}{c}{\text{etc.}} \end{array}$$

is obviously indeterminate when applied to b.

When $\frac{1}{0}$ is applied to a correspondence other than *one*, as ± 8, it is not indeterminate, but simply meaningless, for 8 could not appear at all on the right-hand side of a correspondence of numbers to "*one*."

With the proposition that, if $ab = 0$, either $a = 0$ or $b = 0$, should be compared the statement that if $f^x = 1$, then either $f = 1$, or $x = 0$.

Section 19.—The Meaning of the Square Root of Minus One.

Since the inversor and tensors are distinct operators, permutative to each other, therefore the square root of such a product as

-9 may be written $3\sqrt{-1}$, that is, as the product of the square root of the tensor 9, and of the inversor $-$.

The latter part of the symbol $3\sqrt{-1}$ is, as already said, meaningless, if by $-$ is meant the inversor, because a transposition of only one cycle has no square root.

The language often used, implying that a meaning exists for $\sqrt{-1}$ is, therefore, not free from objections; one is, that it is not the square root of the inversor, but of a somewhat different operator, for which a meaning is found, and, for that operator, it is convenient to retain the sign $-$.

Although a transposition of but one cycle has no square root, transpositions of an even number of cycles have square roots. For example, by actually squaring the cycle of four operands,

$$\begin{array}{cc} a & b \\ b & c \\ c & d \\ d & a \end{array}$$

it will be found that its square is the transposition of *two* cycles,

$$\begin{array}{cc} a & c \\ c & a \\ b & d \\ d & b \end{array}$$

It may be noticed that the inverse of the above cycle of four operands has the same square, so that a transposition of two cycles has two square roots, one of which is the inverse of the other.

The essence of the common explanation of $\sqrt{-1}$ consists in making the symbol $-$ a transposition of two symbols, instead of only one. And this is equivalent to supposing that the correspondences to which the inversor applies are not simply direct and inverse, but direct and inverse of two kinds.

Suppose, for example, that we have two addition tables, one for apples, the other for oranges. Then each table supplies positive and negative correspondences, and therefore we may regard the symbol $-$ as representing a transposition of two cycles, instead of only one.

It now has a square root, which, if the correspondences are distinguished by the letters A and O, may be tabulated as

$+A$	$+O$
$+O$	$-A$
$-A$	$-O$
$-O$	$+A$

or, of course, as the inverse of this cycle of four operands.

It must be observed that the above square root of the inversor is, like the inversor itself, not really a correspondence of operands to operands, for example, of apples to oranges, but one of apple functions to orange functions; and, further, that it is no more "imaginary" than is the inversor.

If A and O, instead of referring to apples and oranges, distinguish vectors along two different lines, then it will be seen that $\sqrt{-1}$ is the correspondence of a vector on one line to a vector on the other. If the vectors have a common origin in the intersection of the two lines at right angles, then $\sqrt{-1}$ may be described as an operator which turns a vector through 90°, while its inverse turns the vector through 90° in the opposite direction.

If $\sqrt{-1}$, as tabulated above, is multiplied by the inversor, their product is the other form of $\sqrt{-1}$; this naturally suggests distinguishing the two forms as $+\sqrt{-1}$ and $-\sqrt{-1}$, but it is a matter of indifference which form has the positive sign.

By actual multiplication, it will be found that the square of $+\sqrt{-1}$ is the inversor, that its cube is $-\sqrt{-1}$, and its fourth power is one, which is denoted by $+$.

We have then four symbols to represent successive powers of a cycle of four operands, and the question suggests itself, cannot the same symbols be employed to represent the operands themselves, in place of the four above, $+A$, $+O$, $-A$, $-O$, just as $+$ and $-$ represent not only one and the inversor, but also distinguish direct from inverse?

In other words, cannot the four symbols, $+\sqrt{-1}$, $-$, $-\sqrt{-1}$, $+$, be made to form a Group, so that the functions which correspond to them are the four powers mentioned above? A trial will show that several ways present themselves; for example, it is indifferent which pair of operands is distinguished by $\sqrt{-1}$. In any case, the method of notation secures the familiar feature of

symbols forming a Group, namely, that they act both as operands and as operators on each other, or, in the present instance, both as functions and super-functions.

The above remarks contain the most important part of the explanation of $\sqrt{-1}$, but there are certain developments, to understand which some knowledge of multiple correspondence (*v.* Chap. V.) is required. Historically, the explanations of $\sqrt{-1}$, and of multiple correspondence, were closely connected, but, at the present day, it is perhaps unfortunate that difficulties of two distinct kinds, that of the extended meaning of the inversor, and that of multiple correspondence, are generally introduced to the reader simultaneously.

Summary of Chapter IV.

The importance of the idea of a Group, even in elementary work, is generally acknowledged, but its abstract character has been an objection to its introduction in schools. The increasing prominence given to the idea of functionality encourages, however, the belief that the idea of a Group, also, may in time receive recognition at an earlier stage than now. In Chapter IV. an attempt is made to assist this recognition, by tracing the close connection which exists between the two ideas.

This connection arises from the circumstance that it is natural to pass from a consideration of the correspondence of operands to operands to that of operands to functions, and that a Group may be regarded as a special case of the latter correspondence. Thus the idea of a Group can be developed by gradual stages from those simpler ideas of correspondence which lend themselves to illustration through classificatory models.

As a correspondence of operands to operands is naturally tabulated in a double column, so one of operands to functions having a quantity in common is tabulated as a duplex of symbols, with an extra row along the top, and an extra column on the left-hand side; a form familiar through the ordinary addition and multiplication tables of numbers, and through numerous examples of every-day life. Thus Chapter IV. wears the aspect of a theory of tables of symbols, as Chapter I., of a theory of double columns of symbols. And the methods of representing by symbols a

member of a duplex (Chap. II.) find an application in representing a symbol inside the table in terms of those on top and side.

Various similarities may be traced between the ideas suggested by tables, and those referred to in Chapter I. For example, the meaning already attached to "multiplication" of one-to-one correspondences naturally suggests the use of the word where one-to-two correspondences are concerned. Its utility is, however, less in the second case, for reasons which, at the same time, favour the introduction of the word "substitution" in its place. With a "power" of a one-to-one correspondence may be compared the results of continued substitution of the operands of a one-to-one correspondence by each other, and the inverse operation suggests the question, whether a one-to-two correspondence is, or is not, "associative."

The idea of a *Group* links itself with that of a *twin-class* correspondence, and the *distributive* property with the *permutative* property of two one-to-one correspondences. Just as operands can be made to form an arbitrary class, or an arbitrary correspondence, so, too, they can be made to form an arbitrary Group. Most important of all, the common notation for a "power" of a Group has the same form as that for a product of functions, a circumstance dealt with more fully in the next paragraph.

As pointed out in Chapter I., great importance attaches to equations between products of direct and inverse correspondences. The importance of a Group in connection with such equations lies in the fact that a Group represents a true multiplication table of the functions which correspond to the operands. Hence, the same features of notation which present themselves for a multiplication table forming a Group also present themselves for every Group. For example, the notation

$$+a+b-c$$

represents a *product* of correspondences no less than

$$a \times b \div c,$$

although the operands themselves are added in the one case, and multiplied in the other.

This feature of Groups renders it a matter of minor importance what names are given to the combination of the operands. It will be found in Chapter V. that, in the case of some multiple Groups,

the terms "addition," and "multiplication," are commonly used, in a manner scarcely justified by their definitions, to describe combinations of multiple operands, the point of real importance being simply that these combinations do form Groups.

A consideration of the distributive property naturally suggests the question, what super-functions, or operators acting upon functions, are distributive over a *product* of functions. From the meaning of multiplication it is clear that (in the case of permutative functions) "tensors" and the "inversor" possess this property, and that a negative number used as multiplier is an instance of the use of both operators.

The inversor has no square root, because a transposition of only one cycle has no square root; so that, if $\sqrt{-1}$ is to have a meaning, then the symbol $-$ has to be extended to a transposition of two cycles instead of only one.

Since 0 does not occur among the functions obtained by adding the function "one" to itself, its claim to be regarded as a number, and appear as such in the addition and multiplication tables of numbers, might be disputed. It is true that the correspondence "one" is bound to make its appearance in the notation for addition, because it is the product of a direct and inverse correspondence; and that the symbol for it here, namely, ± 0, has the appearance of implying that there is a number 0, to which "one" corresponds in the table. But it is this feature of the symbol which seems to be practically unavoidable, and not the assumption that a number 0 exists.

The use of ± 0, as an operator on positive and negative numbers, would, however, arise on quite other grounds, namely, as representing an operator which changes a function to one, because $f^{\pm 0} = 1$. Thus ± 0 would appear among numerical operators, *not* because 0 is a number, but because these multipliers play the part of tensors to the functions, $+a$, $-b$, etc. This view of ± 0 explains its effect in changing $+a$ to ± 0, that is to say, to "one," and also the meaningless character of the inverse operation, or "dividing by zero."

CHAPTER V.

MULTIPLE CORRESPONDENCE.

Section 1.—Meaning of Multiple Correspondence.

By multiple correspondence is meant a correspondence to each other of the members of multiplexes, instead of those of single classes. Such a correspondence is no less a one-to-one correspondence than those considered in Chapter I., since, in a multiple one-to-one correspondence, each member on the one side corresponds to some member on the other. It is merely the fact, that the members of each quantity form a multiplex, which provides an extra feature of importance. The remarks already made upon the correspondence of quantities can thus be reconsidered with this added complication. For example, the one-to-two correspondences of quantities which are multiplexes naturally present themselves for consideration on the same grounds as the one-to-two correspondence of single quantities (Chap. IV.).

Multiple correspondence follows from the union of the ideas of multiplex and of correspondence, just as the union of the ideas of multiplex and of order was, in Chapter III., assumed to give the idea of a space. A chacteristic of a multiple quantity is, that its operands differ from each other in several respects, instead of only one. If they differ in two respects, the quantity forms a duplex, if in three, a triplex, and so on. A one-to-one correspondence between the operands of two such quantities is, as already said, a multiple one-to-one correspondence, and a one-to-n correspondence between several such quantities is a multiple one-to n correspondence.

The operands may be called multiple operands, and each of them wears the aspect of a set of ordinary or single operands. For example, an operand in a duplex correspondence is like a set of

two ordinary operands, and an operand in a triplex correspondence like a set of three.

The simplest illustration of a multiple correspondence is provided by taking models of two duplexes, and placing the members of the one above those of the other to which they are assumed to correspond. For example, if the duplexes are two colour-shape duplexes, then the correspondence of one operand to another is that of one colour and shape together to another colour and shape.

It is not necessary to suppose that the colours and shapes of the one duplex appear also in the other. If they do, then the correspondence is a multiple twin-class one.

If two duplex correspondences have a duplex quantity in common, then the correspondence of the remaining two quantities to each other is the product of the two factor correspondences.

The distribution of the operands of the one duplex relative to those of the other suggests a method of classifying such duplex correspondences. It suggests dividing them into two kinds, according to whether the states of a respect in one duplex correspond to only one, or to both of the respects of the other. For example, in the duplex correspondence above, the colours of one duplex may correspond to the colours alone of the other, or to both colours and shapes of the other. If to the colours alone, then we have a correspondence not only of the operands of one duplex to those of the other, but also of the respects of the one to those of the other.

The above illustration could, in imagination, be extended to triple or quadruple correspondences, by supposing that the members of, say, two model triplexes, or two model quadruplexes, which correspond, are placed in contact with each other.

Another illustration of multiple correspondence is furnished by the correspondence to each other of points in a space, for, with the idea of space adopted in Chapter III., a point in a space is essentially a multiple operand. Points differ from each other in each of those ordered respects which constitute the dimensions of the space. Thus, a point in a duplex space may be regarded as a set of two properties, a point in a triplex space as a set of three properties. Therefore a correspondence of points in a duplex space is a duplex multiple correspondence, one of points in a triplex space is a triplex multiple correspondence.

Something similar is found in Cartesian Geometry, for a one-to-one correspondence of points in a plane is there a correspondence of one set of pairs of co-ordinates to another set of pairs of co-ordinates; and a correspondence of points in three-dimensional space is a correspondence of one set of triplets of co-ordinates to another set of triplets of co-ordinates.

Since by a vector is meant a correspondence of points, obtained as described in Chapter I., Section 5, therefore a vector in a duplex space may be looked upon as a duplex multiple correspondence, and a vector in a triplex space as a triplex multiple correspondence. As the same operands appear in both quantities, we have, in this illustration, a twin-class multiple correspondence.

These illustrations of multiple correspondence show how terms like twin-class, product, direct and inverse, etc., have a meaning for multiple as well as for single correspondences. They also show that new ways of classifying multiple correspondences suggest themselves in addition to those of single correspondences.

Section 2.—Notation for Multiple Operands.

In multiple, as in single, correspondence, in order to be clear where numbers are concerned, it seems advisable to begin with correspondences unconnected with numbers. In the case of single correspondence, we began with such instances as the correspondence of English to French words in a dictionary; in the first of the multiple correspondences referred to above, the multiple operands are pieces of cardboard, differing from each other in colour and shape.

But, even where numbers are concerned, it forms of course no essential feature of either a single or multiple correspondence that it should be capable of numerical calculation. A multiple correspondence may be tabulated as experimentally observed relations, just as would be the correspondence of, say, a boy's height to his age, and the problem of finding formulae which fit the observed facts is a different one.

To tabulate a multiple correspondence involves a consideration of how to represent by symbols the multiple operands. Since they are members of a multiplex, they may be represented by one of the ways described in Chapter II. for representing things in a

multiplex. For example, an operand can be represented by a compound symbol, the parts of which are written in a definite order to show the respect to which each refers; or, the symbols of some alphabet can be combined with those of the parts of the compound symbol to show the respects to which the parts refer, and, in that case, the order in which the symbols of the parts is written becomes a matter of indifference.

Section 3.—Notations for Multiple Correspondence.

By Multiple Algebra * is meant the study of the notations for multiple correspondence. The Algebra of a correspondence of duplexes is sometimes called double Algebra, that of a correspondence of triplexes triple Algebra, and so on. Thus, in a double Algebra each operand can be regarded as a pair of operands, in a triple Algebra, as a triplet of operands.

Since, in a Multiple Algebra, we are dealing with the correspondence of several multiple quantities, it may be convenient not only to distinguish the respects of a multiple quantity from each other, but also those of one multiple quantity from those of another. For example, if a duplex one-to-two correspondence is in question, then two letters, as i, j, may be used to distinguish the variables of one duplex, p, q, for another, and l, m, for the third duplex, which has a one-to-two correspondence to the other two.

Each of the variables distinguished by l, m, has some correspondence, either experimentally observed, or one which can be calculated, to the variables distinguished by i, j, p, q. Thus a duplex one-to-n correspondence is defined by two separate equations, a triplex by three, and so on. The plan of using separate distinguishing symbols for the different multiplexes facilitates the writing of these equations, as is shown below, but, generally, the same distinguishing symbols are used for all the quantities, and the mode of combination of the quantities is defined in a way described later.

It will be seen that a knowledge of the meaning of a multiplex, and of the one-to-n correspondence of operands, is all that is

* An historical account of Multiple Algebra will be found in the "Address on Multiple Algebra," by J. W. Gibbs. Vol. II. of *Collected Works of J. W. Gibbs.*

absolutely necessary to understand the meaning of a multiple correspondence. And a reference to the equations in the next Section will show that a knowledge of how to represent things in a multiplex, and of how to represent different types of one-to-n correspondence of operands, is all that is necessary, in order to represent by symbols various types of multiple correspondence.

Section 4.—Multiple Numbers.

Suppose that the multiple quantities with which we are concerned are number duplexes, like the duplex of vessels containing apples and oranges referred to in Chapter III., as an elementary illustration of a "space."

The respects of the duplexes are distinguished by letters, as i, j, or p, q (which we will suppose to be distinct for each duplex), and each multiple operand is a pair of numbers forming a "multiple" or "complex" number.

The different correspondences, such as addition and multiplication, between numbers, which have been referred to in Chapter IV., now suggest different multiple correspondences of the above multiple operands. In accordance with what has been said, each multiple correspondence may be defined by a pair of equations.

It is convenient to represent these separate one-to-n correspondences, occurring in a multiple correspondence, by employing for their notation the same symbols (such as i, j, p, q, above) which distinguish the respects, and which, therefore, are now used like variables in a formula. For example, the pairs of equations below represent each some duplex one-to-one, or one-to-two correspondence, suggested by the single correspondences already studied (Chaps. I. and IV.). The series A consists of multiple one-to-one correspondences, and the series B of multiple one-to-two correspondences.

A.

(1) $l=i^2$
$m=j^2$

(2) $l=3i$
$m=5j$

(3) $l=i+7$
$m=j+3$

(4) $l=i+j$
$m=i\times j$

B.

(1) $l=i+p$
$m=j+q$

(2) $l=i\times p$
$m=j\times q$

(3) $l=i\div p$
$m=j\div q$

(4) $l=i+p$
$m=j\times p$

(5) $l=i^2+p^2$
$m=j^2+q^2$

(6) $l=i+p+j$
$m=p+j+q.$

Given one of these correspondences, for example, B (4), then, for any two pairs of numbers which play the part of i, j, and p, q, the reader can calculate the third pair, l, m.

Examples of the one-to-one correspondence of multiple numbers are afforded by the multiplication of an amount expressed in several units, as pounds, shillings, pence, or yards, feet, inches. If such an expression is multiplied or divided by some number, as 4, then the result is another multiple number, and the correspondence to the first expression is a multiple correspondence. The symbols, £, s. d., play a part similar to i, j, or p, q, above, in marking the different respects of the multiplex, but, in this instance, the same distinguishing symbols are used for both multiple quantities. In Arithmetical work, the necessity of learning and applying the relations between the different units naturally overshadows in importance the question of the type of multiple correspondence.

The multiplication of a number expressed in the Arabic notation is a similar instance. Here the order in which the symbols, the digits, are written, is made to show the respect to which each refers, and therefore extra symbols, like i, j, or £, s. d., are not necessary.

If the separate numbers in a multiple number can be positive or negative, then an obvious multiple one-to-one correspondence is the operator which changes the sign of each separate number, and, to that operator it is natural to give the symbol $-$.

If used in this way, $-$ can be a transposition of more than one cycle, and, therefore, as pointed out in Chapter IV., can have a square root. For example, if we are dealing with a duplex number, one cycle is the correspondence of $+ai+bj$ to $-ai-bj$, the other the correspondence of $+ai-bj$ to $-ai+bj$. Therefore, with this meaning, $-$ has a square root which might be tabulated as

$$\begin{array}{ll} +ai+bj & +ai-bj \\ +ai-bj & -ai-bj \\ -ai-bj & -ai+bj \\ -ai+bj & +ai+bj \end{array}$$

It may be noticed that this necessary condition for $-$ to have a square root may be satisfied by correspondences into which

numerical changes enter as well as mere change of sign; for example,

$$\begin{array}{ll} +ai+bj & +ci-dj \\ +ci-dj & -ai-bj \\ -ai-bj & -ci+dj \\ -ci+dj & +ai+bj \end{array}$$

As transpositions of this kind were the first examples of the extended meaning of the inversor to be studied, there was naturally a close connection between the explanation of $\sqrt{-1}$ and the theory of multiple Algebra.

In regard to the one-to-two correspondences of series B, it will be noticed what a variety are suggested by even the few one-to-two correspondences of numbers employed, and, hence, the utter insufficiency of words like "addition" and "multiplication" to describe these multiple correspondences. Example B(1) is the most important, and, to this, it is usual to give the term "addition," but it must be observed there is something arbitrary in this choice, and that the use of "addition" and "multiplication" in regard to single correspondences affords no satisfactory guide to their customary use for multiple one-to-two correspondences. From the present standpoint, a correct use of "multiplication" and "addition" is suggested in the same manner for both single and multiple correspondences by paying attention to correspondences having a quantity in common.

Illustrations of B(1) are afforded by the addition of sums in pounds, shillings, pence, or yards, feet, inches, etc., or of numbers expressed in the Arabic notation.

Another example is given by the multiplication of vectors. To define a vector in any space, we must give the relative positions of two corresponding points, as regards each of the dimensions separately. Thus, a vector in a duplex space is defined by a set of two positive or negative numbers, a vector in a triplex space by a set of three. It has already been shown that a vector is a multiple correspondence, because, with the present meaning of "space," a point is a multiple operand; it is now seen that a vector may also itself play the part of a multiple operand, namely a multiple number. Thus, any correspondence of multiple numbers also represents a correspondence of vectors. In particular, the multiple

number standing for a product of two vectors is the "sum" (in the sense of B (1)) of the multiple numbers standing for the factors.

Section 5.—Some Illustrations of Multiple Correspondence.

1. *In Chemistry.*—From the nature of multiple quantities, their correspondence must be met with frequently in the different branches of science. Broadly speaking, such a correspondence arises when we can trace a correspondence not only of one whole to another, but also of parts of the one whole to parts of the other.

The correspondence of two bodies which enter into chemical combination provides an example.

The physical constants of a body may be regarded as one operand of a multiple quantity. If two bodies combine, the relation of their constants to those of the compound, is a multiple one-to-two correspondence, which it is one of the chief aims of physical chemistry to investigate.

2. *In Heredity.*—A person is characterised by a set of data, such as his weight, height, colour of hair, sensitiveness to various stimuli, and so forth. The same data may be ascertained for each of his parents, and the relation of the three sets of data is a multiple one-to-two correspondence, which is investigated, with similar problems, by students of heredity.

Section 6.—"Forms" in Elementary Algebra.

From what has been said, it will be seen that multiple quantities occur in Algebra, when we are concerned, not with single variables, but with expressions containing variables, and when these expressions form a multiplex.

For example, expressions containing two variables may form a duplex, and expressions containing three a triplex, and so on, provided that other symbols exist in the expressions, and play the part of symbols distinguishing the respects of a multiplex.

The word "form" is often used of a multiplex consisting of such expressions. Thus, two expressions are said to be "of the same form" when the same set of distinguishing symbols occurs in both. They agree in having the same set of distinguishing

symbols, but may differ from each other as regards any or all of the variables.

A multiple one-to-two correspondence arises when, by combining in some way two expressions of the same form, a third expression of that form is arrived at.

Examples of Forms.

For example, a difference is a form, because differences can be classified by the value of the positive number, and cross-classified by the value of the negative number, and so form a duplex.

The symbols + and − play a part similar to p, q, or i, j, or £, s. d., in marking the respects of the multiplex.

A fraction is a form, because fractions can be classified by the value of the numerator, and cross-classified by the value of the denominator. If the symbols × and ÷ are used to distinguish numerator from denominator, then they act similarly to + and − above.

A sum of powers, for example, ax^2+bx, is a form, because such sums can be classified by the values of the different coefficients, a, b, etc. (x remaining unchanged), so that expressions of the form above would provide a duplex.

Section 7.—Correspondences of Forms.

Various arbitrary multiple correspondences between expressions of the same form may be suggested, and represented in the way already described. For example, the correspondence of $x-y$ to x^2-y^3 could be defined by the two equations—

$$l=i^2$$
$$m=j^3.$$

But there are also many naturally occurring multiple correspondences provided by these forms. For example, the sum and the product of two differences have the form of a difference; the sum and the product of two fractions have the form of a fraction. The sum of ax^2+bx and $a^1x^2+b^1x$ is an expression of the same form.

It was shown in Chapter IV. that the four symbols, $+$, $-$, $+\sqrt{-1}$, $-\sqrt{-1}$, form a Group, and can be used indifferently for powers of the super-function $\sqrt{-1}$, or to distinguish the direct and inverse functions which play the part of operands to $\sqrt{-1}$. Such an expression as $a+\sqrt{-1}\,.\,b$ provides another illustration of a form, because such expressions can be classified by the value of a, and cross classified by the value of b. The "addition" and "multiplication" of two such expressions give a multiple correspondence, because the "sum" and "product" are expressions of the same form. As already pointed out, the use of "addition" and "multiplication" as names for these multiple correspondences is open to some objections.

In all these cases the essential thing is, that the distinguishing symbols of two multiple operands which combine must reproduce themselves in the result. In other words, when two numbers occurring in the two multiple operands combine in any way (*e.g.*, by addition or multiplication), then the distinguishing symbols belonging to them must also combine so as to give a symbol belonging to the set of distinguishing symbols. The one-to-two correspondence of the distinguishing symbols is, therefore, a twin-class one, and may be exhibited in a table like those of Chapter IV. For the multiplication of differences, this table would be

$$\begin{array}{c|cc} & + & - \\ \hline + & + & - \\ - & - & + \end{array}$$

Except as a matter of convenience, there is, in general, no good reason why such a twin-class correspondence of distinguishing symbols should be called either "addition" or "multiplication."

In "addition" of multiple numbers, each number in one expression is added to the number in the other which has the same distinguishing symbol, and the two distinguishing symbols may be regarded as combining to give their like in the result. With the common meaning of "addition," it is therefore not necessary to consider the combination of each distinguishing symbol with all the others in turn, as has to be done in "multiplication" as usually understood.

Section 8.—Artificial Forms.

In the latter case, the fundamental condition (that two expressions of the same form must reproduce that form) can clearly be satisfied in a variety of ways. Thus, *artificial "forms"* suggest themselves, which may be made to combine in ways different from each other, but equally entitled to be called "multiplication." By an artificial form is therefore meant a multiple number, the distinguishing symbols of which have arbitrary rules of combination of their own, which may be represented in a twin-class table.

It may be observed that the "meaning" of the distinguishing symbols of an artificial form is, that they have been made the operands of an arbitrary twin-class correspondence, and this is the only meaning they need have. All other explanations in the long run amount to no more than this. Similarly, the meaning of a distinguishing symbol as $\sqrt{-1}$ in a natural form is to be looked for in the manner of its combinations with the other distinguishing symbols of that form. In the case of $\sqrt{-1}$, its explanation in the above sense was given in Chapter IV., and any explanation based upon the results of "multiplying" together expressions of the form $a + \sqrt{-1} \, . \, b$ is, from the present standpoint, likely to be misleading, because of the arbitrary meaning given to "multiplication" (*v.* Chap. V., Sect. 10).

It is a matter of indifference how the separate parts of an artificial form are written, so far as the general character of the combinations of two such forms is concerned. Purely as a matter of convenience they are generally written along a line, so that the form wears the aspect of a sum or difference; for example, $a + ib + jc + kd$ where 1, i, j, and k are four distinguishing symbols. This facilitates the "addition" or "multiplication" of two such forms, but, except for this convenience, the separate parts could equally well be written in, say, a column, or round the circumference of a circle.

Among the possibilities* which present themselves in the combination of the distinguishing symbols, the following should be noted:—

* These possibilities were first considered in detail by B. Peirce in the memoir, "Linear Associative Algebra." (*Amer. Jour. Math.*, Vol. IV.)

1. Since the number of combinations which arise when two forms are "multiplied" is greater than the number of distinguishing symbols, some of the combinations of the latter may give zero.
2. Among the distinguishing symbols may occur *one*, and the resultant of two others may then be simply

$$+1 \text{ or } -1.$$

3. A resultant may be arbitrarily made to include some numerical operator, *e.g.*, the squaring tensor, so that, when the two numbers have been combined, the resulting number has to be squared.
4. The combination may, or may not, be permutative.

Section 9.—Multiple Groups.

The step from one-to-one correspondence of operands (Chap. I.) to the correspondence of operands to functions (Chap. IV.) suggests passing from one-to-one multiple correspondence to the correspondence of multiple operands to multiple functions.

Putting aside the possible case, that the multiple functions are in no way connected, then the next case is, that they may be connected by having a quantity in common. From this we may pass to the consideration of one-to-two correspondences of three multiple quantities, each consisting of the same multiple operands. Examples of these correspondences have already been considered, in the case of both naturally occurring, and artificial, correspondences of multiple numbers.

As in the case of single correspondence, the question suggests itself whether a given twin-class table is, or is not, *associative*, and, in the event of two tables consisting of the same multiple operands, whether one table is *distributive* to the other. If a multiple table is finite and associative it forms a multiple Group.

Section 10.—The Common Use of "Addition" and "Multiplication."

A multiple Group, like a single one, is a true multiplication table of the functions which correspond to the operands. Therefore, for any such Group, we must expect to meet with the notation

appropriate to products of direct and inverse functions; that is to say, with three symbols, such as $+$, $-$, and ± 0, which serve the purpose of distinguishing direct from inverse, and of representing the correspondence *one*. In the case of a multiple Group, the last correspondence will, of course, be a multiple function, where each multiple operand on the one side corresponds to the same operand on the other. As regards the combination of the multiple operands themselves, there is no satisfactory reason for calling it either addition or multiplication. To the combination of the type B (1) it is, however, as already said, usual to give the term "addition." Although scarcely justifiable, it has obtained the sanction of long custom, and has a wide range of employment in multiple Algebra and its applications.

It might be supposed that, with such a use of "addition," the word multiplication would be applied only to the combination defined (in the case of a duplex correspondence) by the two equations,

$$l = i \times p$$
$$m = j \times q.$$

Actually, however, "product" is used in many ways in multiple Algebra, and almost the only guiding principle is, that "product" is so defined that "multiplication" is distributive as regards the "addition" referred to above.

With regard to the "forms" referred to as examples of multiple correspondence in Algebra, although "multiplication" of two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, has the meaning we should expect, yet "addition" of fractions does not mean the multiple correspondence referred to above.

Also, although "addition" of two differences $(x-y)$ and $(x'-y')$ has the common meaning, their "multiplication" gives a multiple correspondence which might be defined by the two equations,

$$l = ip + jq$$
$$m = iq + jp.$$

The "multiplication" of two "imaginary numbers" $(a + \sqrt{-1}\,.\,b)$ and $(a' + \sqrt{-1}\,.\,b')$ presents a third variety, represented by the two equations,

$$l = ip - jq$$
$$m = iq + jp.$$

In considering the last kind of "multiplication," there is some convenience in making the order in which two numbers are written show to which respect of the duplex each belongs, instead of employing the symbol $\sqrt{-1}$. For example, we may consider the effect of "multiplying" by the "number-couple" $(-1, 0)$. The effect is to change the sign of each term in the other factor; for example, the couple $(3, 2)$ is converted into $(-3, -2)$ so that $(-1, 0)$ plays the part of the inversor.

A double application of the couple $(0, 1)$, or of $(0, -1)$, gives the same effect, and therefore it is usual to regard these couples as the explanation, or meaning, of $\sqrt{-1}$.

It will be seen, however, that this explanation is open to two objections. Firstly, that, as pointed out in Chapter IV., and in Chapter V., Section 4, $-$ has to be made a transposition of *two* cycles; and secondly, that it is the multiple function *corresponding to* $(0, 1)$ which, when squared, gives this transposition, but we are not, strictly speaking, entitled to call the combination of the couples themselves "multiplication." All that can be said of the couples is, that they form a Group, which is distributive as regards the other Group called "addition."

Section 11.—Meaning of the Cube Root of One.

The same objections which can be raised to the description of the number couple $(0, 1)$ as a "*square root*" of -1, also apply to the application of the term "*cube root of one*" to the expressions $\frac{-1 \pm \sqrt{-3}}{2}$. As already said, the combination of the ordinary "imaginary numbers" $a + \sqrt{-1} \,.\, b$ ought not, strictly speaking, to be called *multiplication*, and therefore the fact that $\frac{-1 + \sqrt{-3}}{2}$, when combined twice with itself gives the number couple $(1, 0)$, does not entitle it to be called a cube root of one. The only unobjectionable statement it is possible to make as to the cube root of one is, to say that it must be a *treposition*, just as all that can be said of the square root of one is, that it must be a transposition, or, of course, one itself.

Now, if we tabulate a treposition, for example,

$$\begin{array}{cc} a & b \\ b & c \\ c & a \end{array}$$

it will be clear that its *inverse* is also a cube root of one. If the treposition and its inverse are denoted by a and β, it can be seen, by actual tabulation, that $a^2 = \beta$ and $\beta^2 = a$, while, as already said, $a^{-1} = \beta$ and $\beta^{-1} = a$.

The combinations of imaginaries to which it is usual to give the term multiplication form a "Group" and represent, therefore, a true multiplication table of the functions *corresponding to* the imaginaries.

Thus, the *function belonging to* $\dfrac{-1+\sqrt{-3}}{2}$ in the table must be a treposition, since its cube, in the correct sense, is "*one.*"

It is for this reason that we find the "square" of $\dfrac{-1+\sqrt{-3}}{2}$ is $\dfrac{-1-\sqrt{-3}}{2}$ and the "square" of the latter is $\dfrac{-1+\sqrt{-3}}{2}$, while again $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^{-1}$ is $\dfrac{-1-\sqrt{-3}}{2}$.

Since the customary use of the term "*addition,*" where imaginaries are concerned, is as much open to objection as is that of "*multiplication,*" the statement that "the sum of the cube roots of one is zero" is doubly misleading, though of course quite correct when properly interpreted.

Section 12.—Representation of Multiple Operands by Single Symbols.

If multiple operands form a Group, then symbolic statements of the relations between them take the form of equations between products of direct and inverse correspondences, as shown in Chapter IV. So far as such equations are concerned, the multiple operands may be represented by single symbols. This use recalls the fact that a multiple correspondence is one of wholes to wholes, in spite of the circumstance that we can trace a minor correspondence between the parts of one whole and those of another.

Suppose that the same multiple operands occur in two permutative Groups, one of which is distributive as regards the other. Then the equations in which these operands, represented by single symbols, occur, may be given the same aspect as those obtained for numbers in ordinary elementary Algebra, by using the symbols $+$, $-$, and ± 0 for the one Group, and $\times$, $\div$, and 1 for the Group which is distributive as regards it. For example, the equations of ordinary Algebra still hold if the variables are replaced by "imaginary numbers" of the form $a + \sqrt{-1} \, . \, b$, and an expression defining a numerical function also defines a "function of a complex variable." It must be remembered, however, that the initial definition of "sum" and "product" is of importance in regard to the suitability of the language commonly used in describing such functions. For example, the numerical function $\cos x$ is defined by the series

$$1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \text{ etc.}$$

But if x is "imaginary," then the signs $+$ and $-$ no longer, strictly speaking, refer to addition, nor is x^2 really a product of x by x.

Section 13.—Case where a Multiple Operand is a Multiplex.

In the sections above, by a multiple operand, has been understood a set of ordinary or single operands. The line of thought followed in this book suggests the question, may not a multiple operand instead of being merely a class of single operands be a multiplex of them?

The representation of multiple operands by single symbols discussed in the last section leads to an answer to this question. For, as already shown, in the section on "forms" in elementary Algebra, correspondences of multiple numbers are suggested by the notation adopted. Therefore, as the same notation is preserved by the algebras of multiple operands (for example of "imaginary numbers"), it follows that we must expect to meet with forms, such as $ai + bj$, or $ai + bj + ck$, etc., where a and b are multiple operands. For example, if a and b are duplex operands, then $ai + bj$ is a duplex of four operands, instead of being merely a set of two operands.

It is to be observed, too, that wherever the notation employed suggests that we are dealing with single operands, we can always push the inquiry a stage further, by supposing that they are in fact multiple operands. When, in addition, one considers the immense variety of artificial forms and correspondences which suggest themselves, it is clear that the subject could easily be developed far beyond the bounds of any usefulness at present attaching to it,

Perhaps the only excuse for referring to the matter here is, that since a correspondence of numbers in Arabic notation is a multiple correspondence, therefore, such a correspondence is, so to speak, a secondary one to any multiple correspondence suggested by ordinary forms, as $a + \sqrt{-1} . b$. From this point of view, the multiple operand (when we are dealing with imaginary numbers) is not simply a set of two single operands but a duplex.

Summary of Chapter V.

Elementary Algebra and Geometry present so many examples of multiple correspondence, that it would be unreasonable not to refer to the subject, supposing that the necessary fundamental ideas are understood. But it is precisely in the establishment of these ideas that classificatory models lend most help. For it seems probable that the idea of considering the correspondences of multiplexes would naturally present itself to any one familiar with ordinary correspondence, and with the idea of a multiplex.

Historically, the development of the subject was assisted by attention becoming concentrated upon the multiple correspondences suggested by the form $a + \sqrt{-1} . b$, and some others, and by the promise of practical usefulness offered by Vector Analysis. Following the plan adopted in the preceding Chapters, I have, however, tried to introduce the preliminary notions without using as illustrations the correspondence of numbers, or of geometric points in the ordinary sense.

The usual methods of representing things in a multiplex (*v.* Chap. II.) serve the purpose of representing a multiple operand, and can be employed to represent a multiple number. It is not necessary to commence with such a special form of the latter as $a + \sqrt{-1} . b$.

Similarly, when taken in conjunction with the ordinary notation for addition and multiplication, etc., of numbers, the methods in question prove sufficient to represent symbolically a great many types of correspondence of multiple numbers. Examples of some of them are easily drawn from elementary Algebra and Geometry.

A consideration of the conditions which must be satisfied for two "forms" to produce their like by "addition," or "multiplication," suggests the invention of artificial "forms," which can likewise be "added" and "multiplied." The distinguishing symbols employed in such artificial forms have arbitrary rules of combination of their own.

The meaning of a Group has already been explained in Chapter IV., and, with that knowledge, the question naturally suggests itself, does a given mode of combining forms provide a Group of them; and (if the same multiple operands appear in two Groups) is one Group distributive as regards the other?

A multiple Group, like a single one, is a true multiplication table of the functions which correspond to the operands, but the grounds for calling the combination of the operands themselves either addition or multiplication are, in general, very slight. In such a case as the addition of two sums expressed in pounds, shillings, pence, the reason is, that they could be expressed in pence, and the combination would then be a genuine addition. But this argument for the use of the word "addition" clearly does not apply to the case of two multiple numbers of the form $a + \sqrt{-1} \,.\, b$. In short, vague analogies are usually the only guide to the use of the words "addition" and "multiplication" where multiple correspondences are concerned.

It follows, that some exception may be taken to the common explanation of $\sqrt{-1}$ as a number-couple, and to that of the meaning of the cube roots of one, as well as to the language used to describe some functions of a complex variable. The doubt extends, of course, only to the suitability of some of the language commonly employed, and not to the accuracy of the symbolic processes.

Since it is usual, and perhaps necessary, to begin the study of mathematics with a rather difficult type of multiple Algebra, namely, ordinary Arithmetic, a few words have been said in

Section 13 upon multiple correspondences where the separate parts of a multiple operand are themselves multiple operands. For, since a number expressed in the Arabic notation is itself a multiple operand, it follows that the correspondences of multiple numbers met with in Algebra are really of the kind just mentioned. Each of the letters a and b, in a form such as $ai + bj$, representing a multiple number, is itself a multiple operand, if its value is expressed in the Arabic notation.

INDEX.

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